Preparing for the Final

Please print your name:

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake all quizzes and both midterms (posted with and without solutions).

(c) Do the new third set of practice problems were compiled from the examples from our lectures.

(d) Do the problems below. (Solutions will be posted soon.)

Bonus challenge. Let me know about any typos you spot in our lecture sketches or the posted solutions (surely, there should be some). Any typo that is not yet fixed on our course website by the time you send it to me, is worth a small bonus.

Problem 2. Find the best approximation of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

Solution. The best approximation we are looking for is the orthogonal projection of $f(x)$ onto $\operatorname{span}\{1, \sqrt{x}\}$, where the dot product of functions is

$$\langle f, g\rangle = \int_0^4 f(t)g(t)\mathrm{d}t.$$

To find an orthogonal basis for $\operatorname{span}\{1, \sqrt{x}\}$, following Gram–Schmidt, we compute

$$\sqrt{x} - \begin{pmatrix}\text{projection of}\\ \sqrt{x}\text{ onto }1\end{pmatrix} = \sqrt{x} - \frac{\langle\sqrt{x}, 1\rangle}{\langle 1, 1\rangle}1 = \sqrt{x} - \frac{4}{3}.$$

In the last step, we used that

$$\langle 1, 1\rangle = \int_0^4 1\mathrm{d}t = 4, \quad \langle\sqrt{x}, 1\rangle = \int_0^4 \sqrt{t}\mathrm{d}t = \left[\frac{1}{3/2}t^{3/2}\right]_0^4 = \frac{16}{3}.$$

Hence, $1, \sqrt{x} - \frac{4}{3}$ is an orthogonal basis for $\operatorname{span}\{1, \sqrt{x}\}$.

The orthogonal projection of $f\colon [0, 4] \to \mathbb{R}$ onto $\operatorname{span}\{1, \sqrt{x}\} = \operatorname{span}\left\{1, \sqrt{x} - \frac{4}{3}\right\}$ therefore is

$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle f, \sqrt{x} - \frac{4}{3}\right\rangle}{\left\langle\sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3}\right\rangle}\left(\sqrt{x} - \frac{4}{3}\right) \;=\; \frac{1}{4}\int_0^4 f(t)\mathrm{d}t + \frac{9}{8}\left(\sqrt{x} - \frac{4}{3}\right)\int_0^4 f(t)\left(\sqrt{t} - \frac{4}{3}\right)\mathrm{d}t.$$

Here, we used that

$$\left\langle\sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3}\right\rangle = \int_0^4 \left(\sqrt{t} - \frac{4}{3}\right)^2\mathrm{d}t = \int_0^4 \left(t - \frac{8}{3}\sqrt{t} + \frac{16}{9}\right)\mathrm{d}t = \left[\frac{t^2}{2} - \frac{16}{9}t^{3/2} + \frac{16}{9}t\right]_0^4 = 8 - \frac{128}{9} + \frac{64}{9} = \frac{8}{9}.$$

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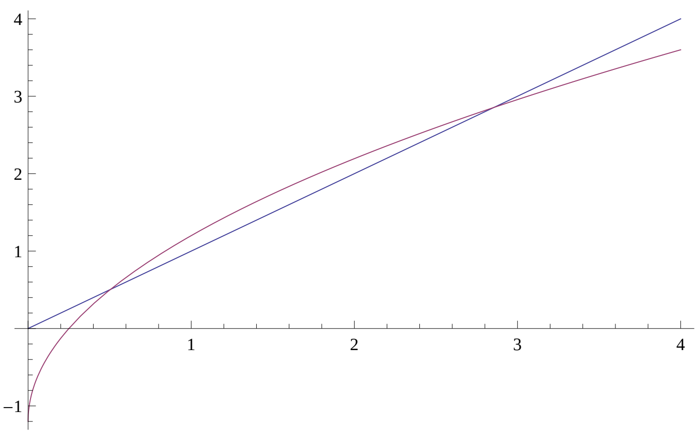
In our case, this best approximation is

$$\begin{aligned} & \frac{1}{4}\int_0^4 t\mathrm{d}t + \frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\int_0^4 t\left(\sqrt{t}-\frac{4}{3}\right)\mathrm{d}t \\ = & \frac{1}{4}\left[\frac{t^2}{2}\right]_0^4 + \frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\left[\frac{2}{5}t^{5/2}-\frac{2}{3}t^2\right]_0^4 = 2+\frac{12}{5}\left(\sqrt{x}-\frac{4}{3}\right) = \frac{12}{5}\sqrt{x}-\frac{6}{5}. \end{aligned}$$

The plot below confirms the quality of this linear approximation:



□

Problem 3. Give a basis for the space of all polynomials $p(x)$ of degree 4 or less such that $p(0)=p(1)$ and $p'(-1)=0$.

Solution. Let us start with the basis $1, x, x^2, x^3, x^4$ for the space of all polynomials $p(x)$ of degree 4 or less.

Then, we can identify the polynomial $p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$.

The condition $p(0)=p(1)$ translates into $a_0=a_0+a_1+a_2+a_3+a_4$, that is, $a_1+a_2+a_3+a_4=0$.

Since $p'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3$, the condition $p'(-1)=0$ translates into $a_1-2a_2+3a_3-4a_4=0$.

In other words, the space of all polynomials $p(x)$ of degree 4 or less such that $p(0)=p(1)$ and $p'(-1)=0$ translates into null(M) with $M=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 \end{bmatrix}$.

Since

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 \end{bmatrix} \overset{R_2-R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 & -5 \end{bmatrix} \overset{-\frac{1}{3}R_2\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \overset{R_1-R_2\Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 & \frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix},$$

the general solution to $M\boldsymbol{x}=\boldsymbol{0}$ is $\begin{bmatrix} s_1 \\ -\frac{5}{3}s_2+\frac{2}{3}s_3 \\ \frac{2}{3}s_2-\frac{5}{3}s_3 \\ s_2 \\ s_3 \end{bmatrix}$. In particular, a basis for null(M) is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2/3 \\ -5/3 \\ 0 \\ 1 \end{bmatrix}$.

The corresponding polynomials are 1, $x^3+\frac{2}{3}x^2-\frac{5}{3}x$ and $x^4-\frac{5}{3}x^2+\frac{2}{3}x$.

Check. Check that these polynomials indeed satisfy $p(0)=p(1)$ and $p'(-1)=0$.

Comment. Let's note that it was to be expected from the beginning that the space is 3-dimensional. The space of all polynomials $p(x)$ of degree 4 or less has dimension 5. Since we impose 2 (independent) conditions, the dimension of our space is $5-2=3$.

□

Problem 4. Consider the edge-node incidence matrix

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

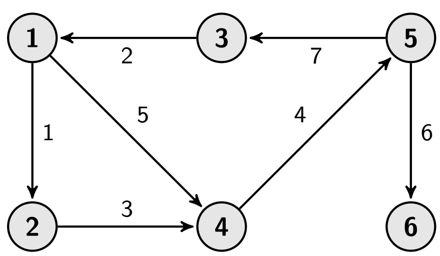
(a) Sketch the directed graph defined by M.

(b) By inspecting the graph, give a basis for null(M).

(c) By inspecting the graph, give a basis for null(M^T).

Solution.

(a)



(b) A basis for null(M) is: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (only one connected component)

(c) A basis for null(M^T) is: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (loops $1 \to 2 \to 4 \to 1$ and $1 \to 4 \to 5 \to 3 \to 1$) □

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