1 Preparing for the Final Exam

- The following problems are just a supplement to the practice problems for Midterm 1 and Midterm 2. The final exam will be comprehensive.
- These problems are taken from the lectures. You can find solutions to all of these in the lecture sketches.
- I will also post additional practice problems before the end of the week.

Example 1. Why is it (strictly speaking) incorrect to say that the eigenvectors of a symmetric matrix are orthogonal?

(nullspace of edge-node incidence matrix)

$\dim \operatorname{null}(A)$ is ______.

In particular, the graph is ______ if and only if $\dim \operatorname{null}(A) = 1$.

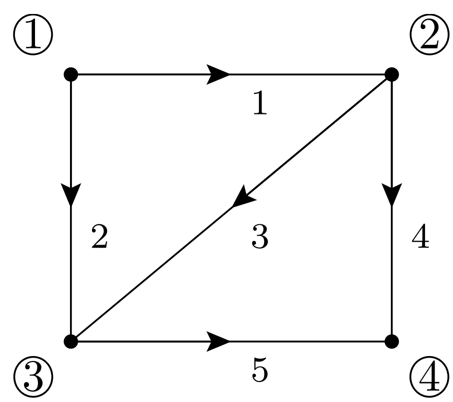
(left nullspace of edge-node incidence matrix)

$\dim \operatorname{null}(A^T)$ is ______.

Example 2. Consider the graph to the right. Determine its edge-node incidence matrix M.

(a) What are $\dim \operatorname{null}(M)$ and $\dim \operatorname{null}(M^T)$?

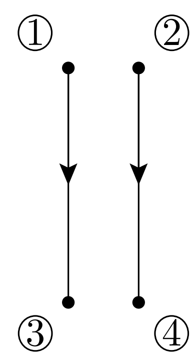
(b) Give bases for $\operatorname{null}(M)$ and $\operatorname{null}(M^T)$.



Example 3. Consider the graph to the right. Determine its edge-node incidence matrix M.

(a) What are $\dim \operatorname{null}(M)$ and $\dim \operatorname{null}(M^T)$?

(b) Give bases for $\operatorname{null}(M)$ and $\operatorname{null}(M^T)$.



Theorem 4. (Euler's formula) In any connected graph, ______.

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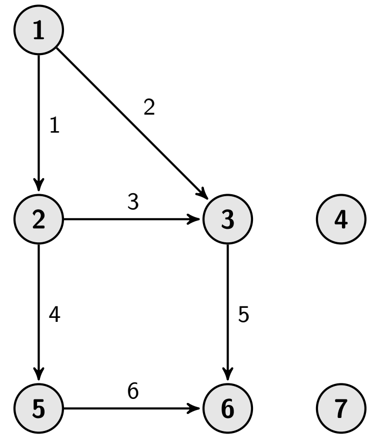
Example 5. Consider the graph to the right. Determine its edge-node incidence matrix M.

(a) What are $\dim\operatorname{null}(M)$ and $\dim\operatorname{null}(M^T)$?

(b) Give bases for $\operatorname{null}(M)$ and $\operatorname{null}(M^T)$.



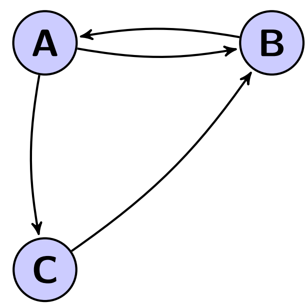
Example 6. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Example 7. Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

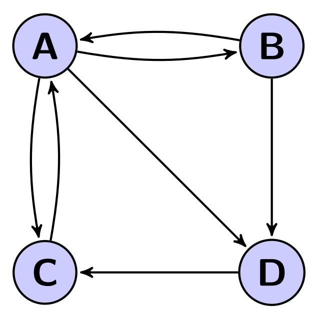
Example 8. Suppose the internet consists of only the three webpages A, B, C which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Example 9. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Example 10. True or false? A^T has the same eigenvalues as A.

Example 11. Which of the following sets are vector spaces?

(a) The set of all functions $\mathbb{R}\to\mathbb{R}$.

(b) The set of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)=0$.

(c) The set of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)=1$.

(d) The set of all functions $f:\mathbb{R}\to\mathbb{R}$ such that f is differentiable.

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Example 12. Which of the following sets are vector spaces? For those that are vector spaces, what is the dimension?

(a) The set of all polynomials (with, say, real coefficients).

(b) The set of all polynomials $p(x)$ such that $p(1)=0$.

(c) The set of all polynomials $p(x)$ such that $p(0)=1$.

(d) The set of all polynomials of degree (exactly) 2.

(e) The set of all polynomials of degree 2 or less.

(f) The set of all polynomials $p(x)$ of degree 2 or less such that $p(3)=0$.

Example 13. Give a basis for the space of all polynomials of degree 3 or less.

Example 14. Give a basis for the space of all polynomials $p(x)$ of degree 2 or less such that $p(3)=0$.

Example 15. Give a basis for the space of all polynomials $p(x)$ of degree 3 or less such that $p(1)=0$ and $p'(1)=0$.

On the space of, say, (piecewise) continuous functions $f: [a,b] \to \mathbb{R}$, it is natural to consider the dot product

$$\langle f, g \rangle = \boxed{}.$$

Example 16. What is the orthogonal projection of $f: [a,b] \to \mathbb{R}$ onto the space of constant functions (that is, $\mathrm{span}\{1\}$)?

Example 17. Find the best approximation of $f(x)=\sqrt{x}$ on the interval $[0,1]$ using a function of the form $y=ax$.

Example 18. Find the best approximation of $f(x)=\sqrt{x}$ on the interval $[0,1]$ using a function of the form $y=a+bx$.

Example 19. Find the best approximation of $f(x)=e^x$ on the interval $[0,1]$ using a function of the form $y=ax+b$.

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