Example 138. Diagonalize, if possible, $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Solution. By expanding by the first row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda).$$

Hence, the eigenvalues are $1, 2, 2$.

- For $\lambda = 1$, the eigenspace $\text{null}\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.
- For $\lambda = 2$, the eigenspace $\text{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Thus, if we choose $P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$, then $A = PDP^{-1}$.

Check that we got it right. We can check this by verifying $AP = PD$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

Comment. Again, there is many other ways to diagonalize the matrix A.

- For instance, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is another basis for the 2-eigenspace (do you see that?).
  Correspondingly, another possible choice is $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$.

Not every matrix is diagonalizable. This is emphasized by the following example:

Example 139. Diagonalize, if possible, $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution. This is a triangular matrix, so we see right away that the eigenvalues are $2, 2$.

The 2-eigenspace $\text{null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ clearly has rank 1, so we see that the 2-eigenspace only has dimension 1.)

The 2×2 matrix A does not have a total of 2 independent eigenvectors. Hence, it is not diagonalizable.

Definition 140. Matrices A and B are **similar** if there is an invertible matrix P such that

$$A = PBP^{-1}.$$

Note. If $A = PBP^{-1}$, then $B = P^{-1}AP$. So the definition works both ways.

Note. So, another way to say that a matrix A can be diagonalized as $A = PDP^{-1}$ is that A is similar to a diagonal matrix.

Comment. The reason that such matrices A and B are called similar is because they represent the same linear transformation in different bases! If A is the matrix with respect to the standard basis, then B is the matrix with respect to the basis given by the columns of P.

Theorem 141. Similar matrices have the same characteristic polynomial.

Proof. Suppose that $A = PBP^{-1}$. Then:

$$\begin{aligned}\det(A - \lambda I) &= \det(PBP^{-1} - \lambda I)\\ &= \det(PBP^{-1} - P\lambda IP^{-1})\\ &= \det(P(B - \lambda I)P^{-1})\\ &= \det(P)\det(B - \lambda I)\det(P^{-1})\\ &= \det(B - \lambda I)\end{aligned}$$

□

Example 142. Do similar matrices have the same eigenvalues?

Solution. By the previous result, similar matrices have the same characteristic polynomial. Hence, they have the same eigenvalues, too (because these are just the roots of the characteristic polynomial).

Example 143. Do similar matrices have the same eigenvectors?

Solution. Similar matrices do not, in general, have the same eigenvectors.

For instance, in our first example today we saw that $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$ are similar.

Obviously, they have the same eigenvalues.

However, the 1-eigenspace of D has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the 2-eigenspace of A has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the eigenspaces are different (the only thing they have in common is the dimension).

Example 144. Find a 2×2 matrix which has 4-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and -1-eigenvector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Are there others?

Solution. Because we have two independent eigenvectors, such a matrix A is diagonalizable as $A = PDP^{-1}$, and we know a possible choice of P and D, namely $P = \begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$. Hence,

$$\begin{aligned}A &= \begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}^{-1}\\ &= \begin{bmatrix} 8 & -5 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -5 \\ -4 & -6 \end{bmatrix}\frac{1}{17}\begin{bmatrix} 6 & -5 \\ 1 & 2 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 43 & -50 \\ -30 & 8 \end{bmatrix},\end{aligned}$$

and this is the unique 2×2 matrix with 4-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and -1-eigenvector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$.