Review. An $n\times n$ matrix A is **diagonalizable** as $A=PDP^{-1}$ if and only if it has n independent eigenvectors.

That's guaranteed to be the case if A has n different eigenvalues. Why?!

Diagonal matrices are very easy to work with.
For instance, it is easy to compute their powers:

Example 136. If $A=\begin{bmatrix} 2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4 \end{bmatrix}$, then what is A^n.

Solution. We compute directly that $A^2=\begin{bmatrix} 2^2 & & \\ & 3^2 & \\ & & 4^2 \end{bmatrix}$. It then becomes obvious that $A^n=\begin{bmatrix} 2^n & & \\ & 3^n & \\ & & 4^n \end{bmatrix}$.

Comment. As done above, it is common to leave zero entries of a matrix blank to emphasize the structure of that matrix.

Example 137. Diagonalize $A=\begin{bmatrix} 0 & -2\\ -4 & 2 \end{bmatrix}$.

Solution. This is the example from last class. A has eigenvalues $-2,4$.
[We can already tell that A is diagonalizable! That's because the 2×2 matrix A will have 2 independent eigenvectors; one for each eigenvalue.]
A -2-eigenvector is $\begin{bmatrix} 1\\ 1 \end{bmatrix}$, and a 4-eigenvector is $\begin{bmatrix} 1\\ -2 \end{bmatrix}$.
[Why is it clear that the eigenspaces have dimension 1?!]
Hence, if we set $P=\begin{bmatrix} 1 & 1\\ 1 & -2 \end{bmatrix}$ (eigenvectors) and $D=\begin{bmatrix} -2 & 0\\ 0 & 4 \end{bmatrix}$ (eigenvalues), then $AP=PD$.
That is, we have the **diagonalization** $A=PDP^{-1}$.

Check that we got it right. We can check this by verifying $AP=PD$:

$$\begin{bmatrix} 0 & -2\\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1\\ 1 & -2 \end{bmatrix}=\begin{bmatrix} 1 & 1\\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & \\ & 4 \end{bmatrix}$$

Alternatives. There is many other ways to diagonalize the matrix A.

- For instance, $P=\begin{bmatrix} 1 & 1\\ -2 & 1 \end{bmatrix}$ and $D=\begin{bmatrix} 4 & 0\\ 0 & -2 \end{bmatrix}$ works just as well.
  (We just changed the order of the eigenvectors.)
- We could also select different eigenvectors: for instance, $\begin{bmatrix} -1\\ 2 \end{bmatrix}$ as our 4-eigenvector.
  In that case, we would get $P=\begin{bmatrix} 1 & -1\\ 1 & -2 \end{bmatrix}$ and $D=\begin{bmatrix} -2 & 0\\ 0 & 4 \end{bmatrix}$, which, again, works just as well.

What's the point? Here is one: note that if $A=PDP^{-1}$, then $A^2=(PDP^{-1})(PDP^{-1})=PD^2P^{-1}$.
Likewise, $A^n=PD^nP^{-1}$.
But D^n is super easy to compute since $\begin{bmatrix} -2 & 0\\ 0 & 4 \end{bmatrix}^n=\begin{bmatrix} (-2)^n & 0\\ 0 & 4^n \end{bmatrix}$.
Using $\begin{bmatrix} 1 & 1\\ 1 & -2 \end{bmatrix}^{-1}=\frac{1}{3}\begin{bmatrix} 2 & 1\\ 1 & -1 \end{bmatrix}$, we therefore have

$$A^n=\begin{bmatrix} 1 & 1\\ 1 & -2 \end{bmatrix}\begin{bmatrix} (-2)^n & 0\\ 0 & 4^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1\\ 1 & -1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} (-2)^n & 4^n\\ (-2)^n & -2\cdot 4^n \end{bmatrix}\begin{bmatrix} 2 & 1\\ 1 & -1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 2\cdot(-2)^n+4^n & (-2)^n-4^n\\ 2\cdot(-2)^n-2\cdot 4^n & (-2)^n+2\cdot 4^n \end{bmatrix}$$

For large n, we see that $A^n\approx\frac{4^n}{3}\begin{bmatrix} 1 & -1\\ -2 & 2 \end{bmatrix}$.

For instance. $A^5=\begin{bmatrix} 320 & -352\\ -704 & 672 \end{bmatrix}\approx\frac{4^5}{3}\begin{bmatrix} 0.938 & -1.031\\ -2.063 & 1.967 \end{bmatrix}$

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Just for fun and curiousity!

Recall that we introduced the **dimension** of a vector space as the number of vectors in a/any basis. In Calculus, on the other hand, you learn about curves (1-dimensional), surfaces (2-dimensional) and solids (3-dimensional).

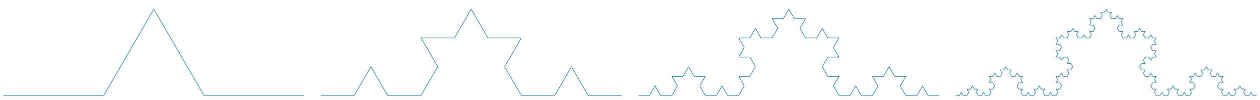
The reason that Linear Algebra is relevant for curved objects like surfaces is that locally these (typically) do look flat (like a plane), so that our tools apply at least locally.

What should a 1.5 dimensional thing look like?

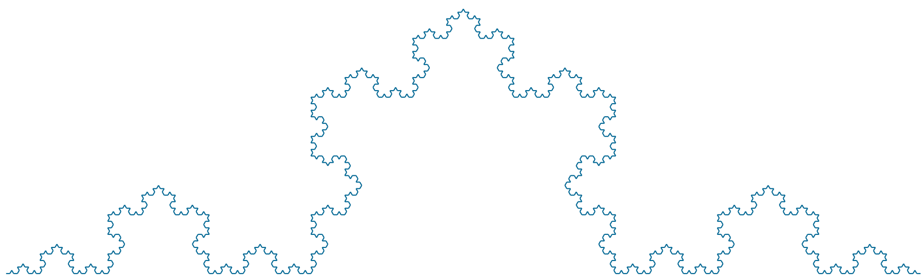
Something between a curve and a surface...

(Note that our linear algebra approach to dimension is not helpful.)

Here is a candidate.



Continuing this process, results in the **Koch snowflake**, a **fractal**:

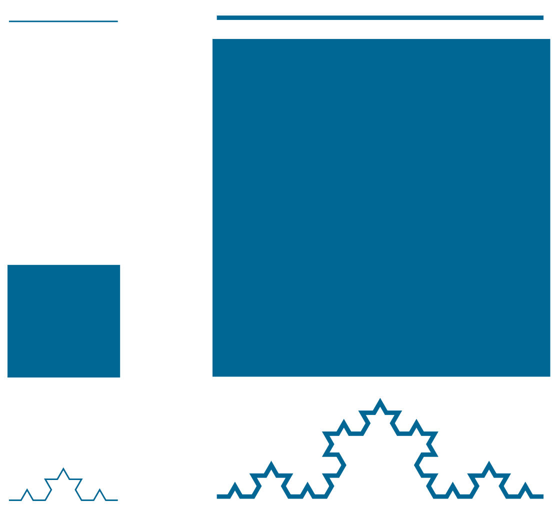


- Its perimeter is infinite!

  Why? At each iteration, the perimeter gets multiplied by $4/3$.

- The table below indicates that its boundary has dimension $\log_3(4) \approx 1.262$!!

the effect of zooming in by a factor of 3



$\times 3$ $\qquad d = 1 = \log_3(3)$

$\times 9$ $\qquad d = 2 = \log_3(9)$

$\times 4$ $\qquad d = \log_3(4) \approx 1.262$

Does this have any practical relevance? Surprisingly, yes!

Have you ever wondered why perimeters of countries are missing from wikipedia? Or, why the coastline of the UK is listed as 11,000 miles by the UK mapping authority but 7,700 miles by the CIA Factbook?

Some of the fun can be found at: https://en.wikipedia.org/wiki/Coastline_paradox

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