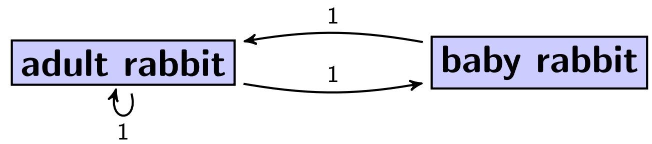
Example 133. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound completely useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let x_t be the number of adult rabbit pairs after t months. Likewise, y_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + y_t \\ x_y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_y \end{bmatrix}.$$

Determine several powers of $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and interpret the values in each column of T^n.

Solution. $T^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $T^3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, $T^4 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $T^5 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$, $T^6 = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$

The first column of T^n equals $T^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the state of 1 adult rabbit pair and 0 baby rabbits. Hence, $T^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ where a (respectively, b) is the number of adult (respectively, baby) rabbit pairs after n months. (Check that this matches the values we obtained in the first column of $T^2, \ldots, T^6$.)

You probably recognize the numbers we are getting: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

These are **Fibonacci numbers**! How fast are they growing?

Did you notice that $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, ...

These ratios approach the **golden ratio** $\varphi = 1.618\ldots$ Where's that coming from?

- If we write F_n for the n-th Fibonacci number, starting with $F_0 = 0$, $F_1 = 1$, then our previous observation translates into $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.
- The eigenvalues of $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio!) and $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$.
- The corresponding eigenvectors are $\boldsymbol{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.
- In terms of the basis of eigenvectors, we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$ with $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$.
- Hence, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = T^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T^n(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = \lambda_1^n c_1 \boldsymbol{v}_1 + \lambda_2^n c_2 \boldsymbol{v}_2$.

  In particular, focusing on the second entry, $F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

  That's **Binet's formula**.
- For large n, $F_n \approx \lambda_1^n c_1$ (because λ_2^n becomes very small).

  In particular, it is very transparent from here that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Comment. In fact, since λ_2 is so small, $F_n = \text{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Advanced comment. Note that the transition matrix connected to the Fibonacci numbers can be obtained directly from the recursive relation $F_{n+1} = F_n + F_{n-1}$, with which the Fibonacci numbers are usually introduced. That's because the recursion is equivalent to $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$.

More importantly, we see that, given any such recursion, we can likewise apply our linear algebra skills.

Diagonalization

Example 134. Diagonalize $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Solution. We have already looked at eigenvalues and eigenvectors of this matrix in Example 124. Since the characteristic polynomial is $-\lambda(2-\lambda) - 8 = \lambda^2 - 2\lambda - 8$, the eigenvalues of A are $-2, 4$.

- A -2-eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a 4-eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
- Note that $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ can be combined as $A\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$.
- In other words, we have $AP = PD$ with $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ (eigenvectors) and $D = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$ (eigenvalues).
- $AP = PD$ is equivalent to $A = PDP^{-1}$. This is called the **diagonalization** of A.
  Fully spelled out (using $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$), it takes the form $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$.

What's the point? Here is one: note that if $A = PDP^{-1}$, then $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$.
Likewise, $A^n = PD^nP^{-1}$.
But D^n is super easy to compute since $\begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}^n = \begin{bmatrix} (-2)^n & 0 \\ 0 & 4^n \end{bmatrix}$. (Continued next time to obtain A^n.)

The key idea of the previous example was to work with respect to a basis of the eigenvectors.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of P are the eigenvectors, and
- the diagonal matrix D has the eigenvalues on the diagonal

Such a diagonalization is possible if and only if A has enough eigenvectors.

Example 135. If a matrix A can be diagonalized as $A = PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. Likewise, $A^n = PD^nP^{-1}$.

The point being that D^n is trivial to compute because D is diagonal.

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