Example 106. Let $A=\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is $\operatorname{rank}(A)$? Find a basis for $\operatorname{col}(A), \operatorname{row}(A), \operatorname{null}(A)$.

Solution. $\operatorname{rank}(A)=2$. Hence, $\dim\operatorname{col}(A)=\dim\operatorname{row}(A)=2$ and $\dim\operatorname{null}(A)=3-2=1$.
The dimension of $\operatorname{null}(A)$ is also called the **nullity** of A.

- A basis for $\operatorname{col}(A)$ is: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- A basis for $\operatorname{row}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- A basis for $\operatorname{null}(A)$ is: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Example 107. Let $A=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What is $\operatorname{rank}(A)$? Find a basis for $\operatorname{col}(A), \operatorname{row}(A), \operatorname{null}(A)$.

Solution.

- A basis for $\operatorname{col}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. In particular, $\operatorname{rank}(A)=1$.
- A basis for $\operatorname{row}(A)$ is: $\begin{bmatrix} 1 \end{bmatrix}$
- $\dim\operatorname{null}(A)=1-1=0$. A basis for $\operatorname{null}(A)$ is: $\{\}$ (the empty set; this basis consists of 0 zero vectors)

Note. Make sure that all of these are evident to you, without computations.

If we insist on computing, the RREF of A is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and, using Theorem 99, we end up with the same bases.

Example 108. Determine a basis for $W=\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\}$.

Note that we have two choices because we can

- use Theorem 99(a) to determine a basis for $W=\operatorname{col}\left(\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}\right)$, or
- use Theorem 99(b) to determine a basis for $W=\operatorname{col}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix}\right)$.

The first option will produce a basis from a subset of the original spanning vectors, while the second option will introduce new vectors (with some zeros). The amount of computation is the same.

Solution. Note that $W=\operatorname{col}(A)$ with $A=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$. Since

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2-R_1\Rightarrow R_2 \\ R_3-R_1\Rightarrow R_3}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \underset{\rightsquigarrow}{R_3-2R_2\Rightarrow R_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for W is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Note. Recall that bases are not at all unique. For instance, now that we know that W is 2-dimensional, we see that any pair of its original spanning vectors would form a basis (because each such pair is linearly independent).

Armin Straub
straub@southalabama.edu

Solution. Note that $W = \mathrm{row}(A)$ with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix}$. Since

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 + R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for W is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Note. In this case, we get a basis that is not taken from the original spanning vectors. Here, we can still see how it is related to the basis we obtained earlier: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Recall that a basis of V is a list of vectors in V, which span V and which are linearly independent. The following is a rephrasing of that:

> Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ in V are a basis of V.
> $\Longleftrightarrow$ Every vector in V can be written uniquely as a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$.

Why? "can be written" because a basis spans V. "uniquely" because basis vectors are linearly independent.

Example 109. Let W be as in the previous example. Is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ in W? Is $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ in W?

Answer using each of the bases we have constructed. If a vector is in W, then write it in terms of the basis.

Solution. We use the basis α: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. [$\alpha$ is just a random name for this basis to distinguish it from the second one]

Since $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{array}\right]$ is inconsistent, $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is not in W.

On the other hand, $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$ is consistent. So, $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ is in W.

$\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Instead of $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, we can write $\begin{bmatrix} 4 \\ -2 \end{bmatrix}_\alpha$. Allows us to work with W as if it was $\mathbb{R}^2$.

Solution. We use the basis β: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

We find, again, that $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is not in W, and that $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ is in W.

Do it! You can proceed exactly as before. But you can also try to exploit the extra 0 in the basis to avoid row operations.

This time, $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Instead of $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, we can now write $\begin{bmatrix} 2 \\ -2 \end{bmatrix}_\beta$.

Armin Straub
straub@southalabama.edu