Example 72. Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R_2 - \frac{c}{a}R_1 \Rightarrow R_2}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc$

Solution. Alternatively, we can expand by the first row (or any other row or column) to get

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\det[\,d\,] - b\det[\,c\,] = ad - bc.$$

Example 73. Compute $\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix}$.

Solution. One option is to proceed by Gaussian elimination, and stop when we arrived at an echelon form:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} \overset{R_4 - 2R_1 \Rightarrow R_4}{=} \begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & -3 \end{vmatrix} \overset{R_4 - R_3 \Rightarrow R_4}{=} \begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot (-4) = -16$$

[Recall that adding a multiple of some row to another one, does not change the determinant!]

Solution. Alternatively, we can expand by the second column:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} = +2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix}$$

We can compute the remaining 3×3 matrix in any way we prefer. One option is to expand by the first column:

$$2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} = 2\left(+1\begin{vmatrix} 2 & 1 \\ 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}\right) = 2(1 \cdot 2 + 2 \cdot (-5)) = -16$$

Important comment. If we elect to do cofactor expansion here, then choosing to expand by the second column is the best choice because this column has more zeros than any other column or row.

Comment. We can also mix & match: start with expansion and then compute $\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix}$ using elimination.

Extra details. Here's the expansion by the second column including the terms that are zero anyway:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} = -0\begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 2 & 8 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

Example 74. Suppose A is a 3×3 matrix with $\det(A) = -2$. What is $\det(10A)$?

Solution. $\det(10A) = 10^3 \cdot (-2) = -2000$

Why? Because A has 3 rows, each of which gets multiplied with 10.

This principle is easiest to see at a trivial example like: $\det\left(10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{vmatrix} = 1000.$

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Example 75. What's **wrong** in the following "calculation" involving $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?!

$$\det(A^{-1}) = \det\left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad - bc}(da - (-b)(-c)) = 1$$

Solution. The corrected calculation is: $\det\left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{(ad - bc)^2}(da - (-b)(-c)) = \frac{1}{ad - bc}$

Remark. If you are still confused about the above mistake: note that $\det\left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 4$ (not 2).

Note. In the corrrected form, we have just shown that $\det(A^{-1}) = \frac{1}{\det(A)}$ for all invertible 2×2 matrices. This relation is actually true in general.

For any $n \times n$ matrices A, B, we have

- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$ (assuming that A is invertible, which is the case precisely if $\det(A) \neq 0$)

The second property is an immediate consequence of the first.
Why? Because $\det(AA^{-1}) = \det(A)\det(A^{-1})$ and $\det(AA^{-1}) = \det(I) = 1$.

Example 76. Let A be an $n \times n$ matrix with $\det(A) = d$. Simplify $\det(A^3)$ and $\det(3A)$.

Solution.

- $\det(A^3) = \det(A \cdot A \cdot A) = \det(A)\det(A)\det(A) = d^3$
- $\det(3A) = 3^n d$ (there are n rows, each scaled by 3)

9 The transpose of a matrix

Definition 77. Interchanging the rows and columns of A produces its **transpose** A^T.

Example 78.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$

A matrix A such that $A^T = A$ is called **symmetric**.

(d) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

[This is useful for typographical reasons, because column vectors take up so much space.]