Example 21. Consider the following linear system:

$$\begin{aligned} x_1 + 6x_2 + x_4 &= 0 \\ 2x_1 + 12x_2 + x_3 - 2x_4 &= 5 \\ x_1 + 6x_2 - x_3 + 11x_4 + x_5 &= 2 \end{aligned}$$

Gaussian elimination:

$$\left[\begin{array}{rrrrr|r} 1 & 6 & 0 & 1 & 0 & 0 \\ 2 & 12 & 1 & -2 & 0 & 5 \\ 1 & 6 & -1 & 11 & 1 & 2 \end{array}\right] \underset{\substack{R_2-2R_1\Rightarrow R_2\\ R_3-R_1\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & -1 & 10 & 1 & 2 \end{array}\right] \underset{R_3+R_2\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 6 & 1 & 7 \end{array}\right]$$

$$\underset{\frac{1}{6}R_3\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right] \underset{\substack{R_1-R_3\Rightarrow R_1\\ R_2+4R_3\Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 6 & 0 & 0 & -\frac{1}{6} & -\frac{7}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{29}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right]$$

- The system is consistent.

  Why? We were able to see that at the moment we had an echelon form. The echelon form had no row of the type $[\,0\;\;0\;\;\ldots\;\;0\,|\,b\,]$ with $b \neq 0$, and so the system is consistent.

- The pivots are located in columns $1, 3, 4$.

- Correspondingly, our free variables are x_2, x_5.

  We set $x_2 = s_1$ and $x_5 = s_2$, where s_1, s_2 can be any numbers (free parameters).

- Solving each equation for the pivot variable, we find that the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} - 6s_1 + \frac{1}{6}s_2 \\ s_1 \\ \frac{29}{3} - \frac{2}{3}s_2 \\ \frac{7}{6} - \frac{1}{6}s_2 \\ s_2 \end{bmatrix}$$

5 Vectors and linear combinations

Example 22. We have already encountered **matrices** such as

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 2 \\ 3 & 2 & -2 & 0 \end{bmatrix}.$$

Each column is what we call a **(column) vector**.

In this example, each column vector has 3 entries and so lies in $\mathbb{R}^3$.

Example 23. A fundamental property of vectors is that vectors of the same kind can be **added** and **scaled**.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}, \qquad 7 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{bmatrix}.$$

Example 24. Let us return to the system we solved at the beginning of this class. Note that we already wrote its general solution as a vector (in $\mathbb{R}^5$). Further note that we can also write it as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} - 6s_1 + \frac{1}{6}s_2 \\ s_1 \\ \frac{29}{3} - \frac{2}{3}s_2 \\ \frac{7}{6} - \frac{1}{6}s_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} \\ 0 \\ \frac{29}{3} \\ \frac{7}{6} \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} \frac{1}{6} \\ 0 \\ -\frac{2}{3} \\ -\frac{1}{6} \\ 1 \end{bmatrix}.$$

Comment. The first vector on the right-hand side is a **particular solution** to our linear system (because that's the solution we get when choosing $s_1 = 0$ and $s_2 = 0$). Plug the other two vectors into our linear system and observe that they solve the equations if the right-hand sides are replaced with 0 (we will call this the **homogeneous system** corresponding to our linear system).

Adding and scaling vectors, the most general thing we can do is:

Definition 25. Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_m$, the vector

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m$$

is a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

The scalars $c_1, \ldots, c_m$ are the **coefficients** or **weights**.

Example 26. Express $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. Clearly, $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 27. Express $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution. We have to find c_1 and c_2 such that

$$c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

This is the same as:

$$\begin{array}{rrcr} c_1 & +2c_2 & = & 3 \\ 3c_1 & +c_2 & = & -1 \end{array}$$

Solving, we find $c_1 = -1$ and $c_2 = 2$.
Indeed,

$$-\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

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