1 Introduction to systems of linear equations

Definition 1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written as

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

for some numbers $b, a_1, a_2, \ldots, a_n$.

Example 2. Which of the following equations are linear?

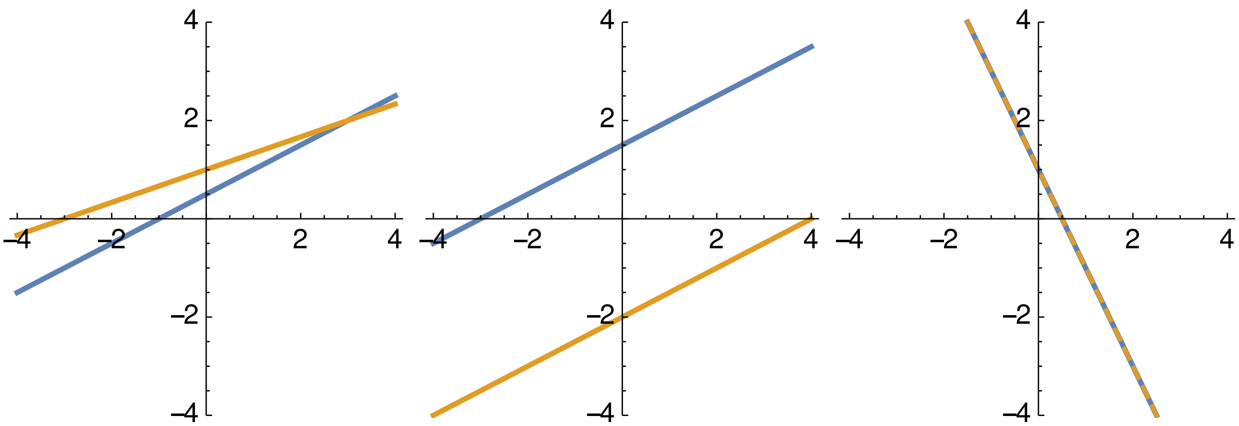
- $4x_1 - 5x_2 + 2 = x_1$ linear: $3x_1 - 5x_2 = -2$
- $4x_1 - 6x_2 = x_1 x_2$ not linear: $x_1 x_2$
- $x_2 = 2\sqrt{x_1} - 7$ not linear: $\sqrt{x_1}$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$ linear: $2x_1 + x_2 - x_3 = 2\sqrt{6}$

Definition 3. A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Example 4. (Two equations in two variables)
In each case, sketch the set of all solutions.

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ -4x_1 & - & 2x_2 & = & -2 \end{array}$$



Theorem 5. A linear system has either

- no solution, or
- one unique solution, or
- infinitely many solutions.

Definition 6. A system is **consistent** if a solution exists.

2 How to solve systems of linear equations

Strategy: replace system with an equivalent system which is easier to solve

Example 7. Let us solve the systems from the previous example:

- $$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \overset{R_2+\frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \quad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ & & x_2 & = & 2 \end{array}$$

  Once in this **triangular** form, we find the solutions by **back-substitution**:

  $$x_2 = 2, \qquad x_1 = 3$$

- $$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \quad \overset{R_2-2R_1 \Rightarrow R_2}{\rightsquigarrow} \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ & & 0 & = & 14 \end{array}$$

  Since $0 \neq 14$, this system is inconsistent. It has no solutions.

- $$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ -4x_1 & - & 2x_2 & = & -2 \end{array} \quad \overset{R_2+2R_1 \Rightarrow R_2}{\rightsquigarrow} \quad \begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ & & 0 & = & 0 \end{array}$$

  In this case, the second equation was redundant (it clearly is a multiple of the first one). To write down solutions to the system, we set $x_2 = s$, where s is a **free parameter** (i.e. it can take any value). Then, we again proceed by back-substitution:

  $$x_2 = s, \qquad x_1 = \frac{1-s}{2}$$

Example 8. The same approach works for more complicated systems.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & - & 2x_2 & - & 6x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \begin{array}{l} \downarrow R_2 - 2R_1 \Rightarrow R_2 \\ R_3 + 4R_1 \Rightarrow R_3 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \begin{array}{l} \downarrow \\ R_3 + \frac{3}{2}R_2 \Rightarrow R_3 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

By back-substitution:

$$x_3 = 3, \qquad x_2 = 16, \qquad x_1 = 29.$$

Note. It is always a good idea to check our answer. Let us check that $(29, 16, 3)$ indeed solves the original system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & - & 2x_2 & - & 6x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{array}{rcrcrcr} 29 & - & 2 \cdot 16 & + & 3 & \overset{\checkmark}{=} & 0 \\ 2 \cdot 29 & - & 2 \cdot 16 & - & 6 \cdot 3 & \overset{\checkmark}{=} & 8 \\ -4 \cdot 29 & + & 5 \cdot 16 & + & 9 \cdot 3 & \overset{\checkmark}{=} & -9 \end{array}$$

Example 9. (Exercise!) Solve the following linear system:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ -x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 5 \end{array}$$

(To stick to our strategy, your first step should be $R_2 + R_1 \Rightarrow R_2$, $R_3 - 2R_1 \Rightarrow R_3$.)

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