Preparing for Midterm #1

MATH 237 — Linear Algebra I
Thursday, Oct 1

Please print your name:

Problem 1. Compute the following, or state why it is not possible to do so:

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^T$

(b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1}$

(c) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$

(d) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

(e) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Solution.

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^T$

(b) Non-square matrices are not invertible.

(c) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{4\cdot 1 - 2\cdot 1}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$

(d) This matrix is not invertible.

(e) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ 4 & 0 & 2 \\ 13 & -1 & 8 \end{bmatrix}$

(f) A 2×3 matrix and a 2×2 matrix cannot be multiplied (in that order). □

Problem 2. Let A be a $p\times q$ matrix and B be an $r\times s$ matrix. Under which condition is $A^T B$ defined?

Solution. A^T is $q\times p$ and B is $r\times s$. Hence, we need $p = r$. □

Problem 3. Decide whether the following statements are true or false.

(a) If A is invertible then the system $A\boldsymbol{x} = \boldsymbol{b}$ always has the same number of solutions.

(b) The homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is always consistent.

(c) In order for A to be invertible, the matrix A has to be square (that is, of shape $n\times n$).

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(d) If A is a 4×3 matrix with 2 pivot columns, then the columns of A are linearly independent.

(e) If A is invertible then the columns of A are linearly independent.

(f) $\boldsymbol{b}$ is in the span of the columns of A if and only if the system $A\boldsymbol{x}=\boldsymbol{b}$ is consistent.

(g) Every matrix can be reduced to echelon form by a sequence of elementary row operations.

(h) The row-reduced echelon form of a matrix is unique.

Solution.

(a) True. $A\boldsymbol{x}=\boldsymbol{b}$ always has exactly one solution.

(b) True. It has the trivial solution $\boldsymbol{x}=\mathbf{0}$.

(c) True.

(d) False. Since A has 3 columns but only 2 pivots, the system $A\boldsymbol{x}=\mathbf{0}$ has a free variable. This means that $A\boldsymbol{x}=\mathbf{0}$ has solutions beside the trivial one, and so the columns of A are linearly dependent.

(e) True, because $A\boldsymbol{x}=\mathbf{0}$ only has the unique (trivial) solution $x=A^{-1}\mathbf{0}=\mathbf{0}$.

(f) True. Keep in mind that $A\boldsymbol{x}$ is a linear combination of the columns of A.

(g) True. And we can do it! All the way to a RREF if necessary.

(h) True. □

Problem 4. We are solving a linear system with 4 equations and 5 unknowns. Which of the following are possible?

(a) The system has no solution.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

Solution. The corresponding matrix A has 4 rows and 5 columns. Since there is only room for 4 pivots, there has to be at least one free variable. Hence, the system cannot have a unique solution. The other two options are possible. □

Problem 5. We are solving a linear system with 5 equations and 5 unknowns. Which of the following are possible?

(a) The system has no solution.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

Solution. All of these are possible. □

Problem 6. For which values of a is the matrix $\begin{bmatrix} 3 & a-6 \\ 3a & -a+6 \end{bmatrix}$ invertible?

Solution. Recall that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is invertible if and only if $ad-bc \neq 0$. Hence, the matrix above is invertible if and only if

$$3(-a+6)-(a-6)(3a)=3(a^2+5a+6)=3(a+2)(a+3)\neq 0.$$

We conclude that the matrix is invertible for all values of a except $a=-2$ and $a=-3$. $\square$

Problem 7. Consider the vectors

$$\boldsymbol{w}=\begin{bmatrix} 2 \\ -4 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_1=\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2=\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Is $\boldsymbol{w}$ in $\operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$? If so, write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$.

(b) Determine whether the vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are linearly independent.

(c) Is $\boldsymbol{v}_3$ in $\operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$?

Solution.

(a) We eliminate!

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{R_3-R_1\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{\substack{\frac{1}{2}R_2\Rightarrow R_2 \\ R_3-\frac{1}{2}R_2\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{R_4+R_3\Rightarrow R_4}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this echelon form, we see that the system is consistent. This means $\boldsymbol{w}$ is in $\operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$.

To write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, we have to solve the system: by back-substitution, we find $x_3=1$, $x_2=-2$ and $x_1=2-x_3=1$. The linear combination is

$$\boldsymbol{w}=\boldsymbol{v}_1-2\boldsymbol{v}_2+\boldsymbol{v}_3.$$

(b) Our computation in the first part (forget about the augmented part) shows that the system $\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]\boldsymbol{x}=\boldsymbol{0}$ has no free variables. Hence, $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are linearly independent.

(c) No. Because $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are linearly independent, it is not possible to write $\boldsymbol{v}_3=x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2$ (because then we would have the dependence relation $x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2-\boldsymbol{v}_3=\boldsymbol{0}$). $\square$

Problem 8. Consider the vectors

$$\boldsymbol{v}_1=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2=\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_3=\begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}.$$

(a) For which value(s) of h is $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(b) For which value(s) of h are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent?

Solution.

(a) We have to determine whether the following system has a solution:

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{array}\right] \underset{\rightsquigarrow}{\overset{-R_2 \Rightarrow R_2}{R_3+R_1 \Rightarrow R_3}} \left[\begin{array}{rr|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & h+1 \end{array}\right] \underset{\rightsquigarrow}{R_3-3R_2 \Rightarrow R_3} \left[\begin{array}{rr|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h+1 \end{array}\right]$$

This system is consistent if and only if $h = -1$. Hence, $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for $h = -1$ only.

(b) Since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are clearly independent (just two vectors that are not multiples of each other), the three vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ can be dependent only if $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (can you see that?). By the first part, this happens for $h = -1$ only. Hence, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent for all $h \neq -1$.

If you didn't notice that, you would have to determine whether the system $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$ has solutions apart from the trivial one. As above (actually the same matrix!)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h+1 \end{bmatrix}$$

and we see that there is 3 pivots (and no free variables) unless $h = -1$. Again, we find that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent for all $h \neq -1$. □

Problem 9. Consider $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Determine A^{-1}.

(b) Using (a), solve $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Solution.

(a)

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2-R_1 \Rightarrow R_2}{R_3-R_1 \Rightarrow R_3}} \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_1+2R_3 \Rightarrow R_1}{\overset{R_2-2R_3 \Rightarrow R_2}{-R_3 \Rightarrow R_3}}} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Hence, $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$.

(b) $\boldsymbol{x} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ □

Problem 10. Consider $B = \begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 2 & 4 & 14 & 12 & -12 & -2 \\ 1 & 2 & 4 & 3 & -2 & 6 \end{bmatrix}$.

(a) Determine the row-reduced echelon form of B.

(b) Use your result in (a) to find the general solution of the linear system:

$$\begin{array}{rrrrrcr} x_1 & +2x_2 & +6x_3 & +5x_4 & -5x_5 & = & 0 \\ 2x_1 & +4x_2 & +14x_3 & +12x_4 & -12x_5 & = & -2 \\ x_1 & +2x_2 & +4x_3 & +3x_4 & -2x_5 & = & 6 \end{array}$$

(c) What is the general solution to the associated homogeneous linear system?

(d) Write down, in vector form, the general solution to $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}$. Also decompose the solution into a particular solution and the solutions to the homogeneous system.

(e) What is the general solution to $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$? (Note that the RHS is also a column of the matrix!)

(f) Are the columns of $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix}$ linearly independent? If not, write down a non-trivial linear combination of the columns, which produces $\mathbf{0}$.

Solution.

(a)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 6 & 5 & -5 & 0 \\ 2 & 4 & 14 & 12 & -12 & -2 \\ 1 & 2 & 4 & 3 & -2 & 6 \end{array}\right] \underset{\substack{R_3 - R_1 \Rightarrow R_3 \\ \rightsquigarrow}}{\overset{R_2 - 2R_1 \Rightarrow R_2}{}} \left[\begin{array}{ccccc|c} 1 & 2 & 6 & 5 & -5 & 0 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & -2 & -2 & 3 & 6 \end{array}\right] \underset{\rightsquigarrow}{R_3 + R_2 \Rightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 6 & 5 & -5 & 0 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\underset{\rightsquigarrow}{R_1 - 3R_2 \Rightarrow R_1} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 1 & 6 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \underset{\substack{R_2 + 2R_3 \Rightarrow R_2 \\ \rightsquigarrow}}{\overset{R_1 - R_3 \Rightarrow R_1}{}} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \underset{\rightsquigarrow}{\tfrac{1}{2}R_2 \Rightarrow R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

(b) x_2 and x_4 are free variables, and we set $x_2 = s_1$ and $x_4 = s_2$. Then, the general solution is

$$\begin{aligned} x_1 &= 2 - 2s_1 + s_2 \\ x_2 &= s_1 \\ x_3 &= 3 - s_2 \\ x_4 &= s_2 \\ x_5 &= 4 \end{aligned}$$

where s_1, s_2 can take any value.

(c) The general solution is

$$\begin{aligned} x_1 &= -2s_1 + s_2 \\ x_2 &= s_1 \\ x_3 &= -s_2 \\ x_4 &= s_2 \\ x_5 &= 0 \end{aligned}$$

where s_1, s_2 can take any value.

(d) This is the same system as in (b). Hence, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 2-2s_1+s_2 \\ s_1 \\ 3-s_2 \\ s_2 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 4 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{general solution to homogenous system}}.$$

(e) Because $[\ 2\ \ 4\ \ 2\ ]^T$ is just the second column, we have an obvious particular solution, namely

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the general solution is

$$\boldsymbol{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{general solution to homogenous system}}.$$

(f) They are definitely linearly dependent, because these are 5 vectors in $\mathbb{R}^3$.

To write down a linear dependence relation, we have to solve the system $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. But that is just the homogeneous system that we have already solved in part (c) and used in the other parts.

We have two very different solutions to the homogeneous equation, namely

$$\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

To be explicit, the corresponding linear dependence relations for the columns of our matrix are

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 14 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -5 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 6 \\ 14 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -5 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

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