Review 132. Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + y \geqslant 0 \right\}$ a vector space?

Solution. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$. Therefore, if W is a vector space, then any vector in $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ has to be in W. However, for instance, $-1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in W. Hence, W is not a vector space.

Note. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + y = 0 \right\}$ is a vector space. (Why?) Geometrically, U is a line. We conclude that W consists of the vectors/points in $\mathbb{R}^2$, which lie above this line. In other words, geometrically, W is a half-plane.

Review. determinants, 2×2 inverses, row operations

Example 133.

- $\begin{vmatrix} 3 & 2 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 7 \end{vmatrix} = 3 \cdot 4 \cdot 7 = 84$
- $\begin{vmatrix} 3 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 7 \end{vmatrix} = 3 \cdot 0 \cdot 7 = 0$
- $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 13 \end{vmatrix} = 0$

  Why? Because we see that the first two columns are multiples of each other. Since the columns are dependent, the matrix is not invertible, and so has determinant 0.

Example 134. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R_2 - 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix} \overset{R_3 - \frac{4}{7}R_2 \Rightarrow R_3}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix} = 1 \cdot (-7) \cdot \left(-\frac{1}{7}\right) = 1$$

Example 135. Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R_2 - \frac{c}{a}R_1 \Rightarrow R_2}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc$$

Example 136. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R_4 - \frac{3}{2}R_3 \Rightarrow R_4}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot \frac{7}{2} = 14$$

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