Example 121. Find a basis for $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$ with $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix}$.

Solution. Our first step is to bring A into RREF (just an echelon form would be enough, but then we would need to back-substitute when solving $A\boldsymbol{x} = \mathbf{0}$ for $\mathrm{null}(A)$):

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix} \underset{\text{do it!}}{\overset{\text{RREF}}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- A basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$. $(\dim \mathrm{col}(A) = 3)$
- A basis for $\mathrm{row}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $(\dim \mathrm{row}(A) = 3)$
- $x_2 = s_1$ and $x_4 = s_2$ are our free variables. The general solution to $A\boldsymbol{x} = \mathbf{0}$ is:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 + 3s_2 \\ s_1 \\ -4s_2 \\ s_2 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{null}(A)$. $(\dim \mathrm{null}(A) = 2)$

Comments.

- Make sure that you understand the reasoning behind the recipes that we are using to obtain these bases!
- Note that $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^3$. Because it is 3-dimensional, we have that $\mathrm{col}(A) = \mathbb{R}^3$.
- Our recipe from Theorem 116 demands that we select the the non-zero rows of the echelon form to get a basis for $\mathrm{row}(A)$. In this particular case, why do the original rows of A also form a basis of $\mathrm{row}(A)$?

Let A be $m \times n$, and let r be the **rank** of A, that is, r is the number of pivots.

- $\dim \mathrm{col}(A) = \dim \mathrm{row}(A) = r$
- $\dim \mathrm{null}(A) = n - r$

Example 122. Find a basis for $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$ with $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix}$.

Solution. For this simple matrix, we can just "see" the following (make sure you do, too!):

- A basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

  Why? Because these two vectors span and are clearly independent.
- A basis for $\mathrm{row}(A)$ is: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

  Why? Again, because these two vectors span and are clearly independent.
- A basis for $\mathrm{null}(A)$ is: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

  Why? We already know that $\mathrm{rank}(A) = 2$. Hence, $\dim \mathrm{null}(A) = 3 - 2 = 1$. Therefore, any non-zero vector in $\mathrm{null}(A)$ will be a basis for $\mathrm{null}(A)$. Clearly, $[\,0 \ \ 0 \ \ 1\,]^T$ is one such vector solving $A\boldsymbol{x} = \mathbf{0}$ (why?).

Armin Straub
straub@southalabama.edu

Example 123. Find a basis for $\operatorname{col}(A), \operatorname{row}(A), \operatorname{null}(A)$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution. For this simple matrix, we can just "see" the following (make sure you do, too!):

- A basis for $\operatorname{col}(A)$ is: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- A basis for $\operatorname{row}(A)$ is: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- A basis for $\operatorname{null}(A)$ is: $\{\}$ (the empty set; this basis contains no vectors)
  Why? We already know that $\operatorname{rank}(A) = 2$. Hence, $\dim \operatorname{null}(A) = 2 - 2 = 0$.
  Therefore, a basis of $\operatorname{null}(A)$ has to consist of 0 vectors.

Every set of linearly independent vectors in V can be extended to a basis of V.

In other words, let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be linearly independent vectors in V. If V has dimension d, then we can find vectors $\boldsymbol{v}_{p+1}, \ldots, \boldsymbol{v}_d$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ is a basis of V.

Example 124. Consider $H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Give a basis for H. What is the dimension of H?

(b) Extend the basis of H to a basis of $\mathbb{R}^3$.

Solution.

(a) The vectors are independent. By definition, they span H.

Therefore, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for H.

In particular, $\dim H = 2$.

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not a basis for $\mathbb{R}^3$. Why?

Because a basis for $\mathbb{R}^3$ needs to contain 3 vectors.

Or, because, for instance, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not in H. (Can you see it?)

So: just add this (or any other) missing vector! [Note that one of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has to be missing.]

By construction, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are independent.

Hence, this automatically is a basis of $\mathbb{R}^3$.

Armin Straub
straub@southalabama.edu