Let us begin with recalling the definition of a basis and rephrasing Theorem 105.

Review. Three equivalent ways to determine whether a set of vectors (from V) is a basis of a vector space V:

- if the vectors span V and are independent (that's the definition of a basis)
- if the vectors are independent and it's the right number (requires us to already know $\dim V$)
- if the vectors span V and it's the right number (requires us to already know $\dim V$)

Example 108.

(a) Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ a basis for $\mathbb{R}^3$?

These are three vectors, which is the right number for a basis of $\mathbb{R}^3$. They form a basis if and only if they are linearly independent. But

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

reveals that the vectors are linearly dependent. Hence, they do not form a basis of $\mathbb{R}^3$.

(b) Is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution. Three vectors span $\mathbb{R}^3$ if and only if they are basis of $\mathbb{R}^3$. But we have just seen that these three vectors are not a basis (because they are not independent). Hence, they do not span all of $\mathbb{R}^3$.

(c) Find a basis for $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Solution. It follows from the dependence relation $3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}$ (go and find it for yourself from the echelon form we already have!) that there is redundancy in the three spanning vectors. In this case, each one can be dropped without affecting the span:

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

In each case, we are left with two spanning vectors, which are clearly independent (because it's two and they are not multiples of each other). Hence, we found three different bases for V:

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a basis for V, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ is a basis for V, and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ is a basis for V.

If $V = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$, then we can always find a basis of V by taking a subset of the spanning vectors.

Theorem 109 tells us that one possibility to do this is to write $V = \operatorname{col}(A)$ where A is the matrix with columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$. Then, compute an echelon form of A. The columns of A (not of the echelon form!) corresponding to the pivot columns form a basis for V.

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Bases for column spaces

- If the columns of A are independent, then (obviously!) a basis of $\mathrm{col}(A)$ is given by (all) the columns of A.
- Recall that the columns of A are independent

  $\Longleftrightarrow$ $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution (namely, $\boldsymbol{x} = 0$),

  $\Longleftrightarrow$ A has no free variables.

Theorem 109. A basis for $\mathrm{col}(A)$ is given by the columns of A (!) which do not correspond to a free variable.

Example 110. Find a basis for $\mathrm{col}(A)$ with $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$.

Solution. That's exactly the same question as the previous one!

Solution. Since

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

[**Warning:** Note that we cannot take the columns of the echelon form to get a basis for our space! Can you see why this would definitely lead to an incorrect answer? (Focus on the zeros in the third entry.)]

Example 111. Find a basis for $\mathrm{col}(A)$ with $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$.

Solution. We compute an echelon form of A:

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are the first and third. Hence, a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

[Again, note that it would be horribly wrong to take columns from the echelon form.]