Example 95. Find the general solution to $\begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution. We eliminate $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ -1 & 0 & 1 & 0 & -1 \end{array}\right] \underset{\text{(do it!)}}{\overset{\text{RREF}}{\rightsquigarrow}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 \end{array}\right]$.

We set $x_3 = s_1$ and $x_4 = s_2$ because these are our free variables. Then, $x_1 = 1 + s_1$ and $x_2 = 1 - s_1 - 2s_2$. So, our general solution to $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \begin{bmatrix} 1+s_1 \\ 1-s_1-2s_2 \\ s_1 \\ s_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} \underbrace{+ s_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{general solution of } A\boldsymbol{x}=\boldsymbol{0}}.$$

where s_1 and s_2 can be any numbers.

Example 96. Proceeding by some unknown technique (or divine inspiration), an alien proposes instead that the general solution to the previous problem is

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + a \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}.$$

Are you both right?

Solution. Short answer: Yes, this is another way to write the same general solution in different form.

First off, we can quickly check that the alien solution is indeed a solution. For that, we check that $[\, 0 \;\; 0 \;\; -1 \;\; 1 \,]^T$ is a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ and that $[\, 1 \;\; 1 \;\; 1 \;\; -1 \,]^T$ and $[\, 2 \;\; 0 \;\; 2 \;\; -1 \,]^T$ are solutions to $A\boldsymbol{x} = \boldsymbol{0}$. Indeed:

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The other thing to be checked is that the alien solution is general (that is, it includes all the solutions and doesn't miss any). A cheap way to do that is to note that the two vectors $[\, 1 \;\; 1 \;\; 1 \;\; -1 \,]^T$ and $[\, 2 \;\; 0 \;\; 2 \;\; -1 \,]^T$ are linearly independent (why?). Hence the alien solution has two genuine degrees of freedom (a choice for a and for b). This is the same number of degrees of freedom as in our solution (which we already know is general).

Vector spaces

Example 97. The previous example implied that

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

We need to understand such collections of vectors better. These are not just sets of vectors but these are **spaces** of vectors.

Armin Straub
straub@southalabama.edu

Definition 98.

- A **(vector) space** is a set V of vectors that can be written as a span.

  That is, $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ for some bunch of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$
- If $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ are linearly independent, then the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ are called a **basis** of V.
- The **dimension** of V is the number of elements in such a basis. (It is always the same.)

Example 99. The following are important spaces associated with an $m \times n$ matrix A.

- $\operatorname{col}(A)$ (the **column space** of A) is the span of the columns of A.

  This is a **subspace** of $\mathbb{R}^m$ (because each column has m entries, and so lives in $\mathbb{R}^m$).
- $\operatorname{row}(A)$ (the **row space** of A) is the span of the rows of A.

  This is a **subspace** of $\mathbb{R}^n$ (because each row has n entries, and so lives in $\mathbb{R}^n$).
- $\operatorname{null}(A)$ (the **null space** of A) is the set of all solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

  This is also a **subspace** of $\mathbb{R}^n$ (because $\boldsymbol{x}$ has to have n entries for $A\boldsymbol{x}$ to be defined).

Example 100. Explain why $\operatorname{null}(A)$ really is a vector space!

Note that $\operatorname{col}(A)$ and $\operatorname{row}(A)$ are defined as spans, so they are definitely spaces.

Solution. This is a consequence of our knowledge on solving the equation $A\boldsymbol{x} = \boldsymbol{0}$. Since $\boldsymbol{x} = \boldsymbol{0}$ is a particular solution, the general solution is always of the form $\boldsymbol{0} + s_1\boldsymbol{w}_1 + s_2\boldsymbol{w}_1 + \ldots$ where s_1, s_2, ... are the values assigned to the free variables. In other words, $\operatorname{null}(A) = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$.

Example 101. It follows from our first two examples today that

$$\operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}\right\}.$$

Example 102. Convince yourself that $\mathbb{R}^3$ is a vector space. What is its dimension?

Solution. Note that $\mathbb{R}^3 = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}\right\}$. Writing $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we see that $\mathbb{R}^3$ consists of all the linear combinations of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. In other words, $\mathbb{R}^3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

This demonstrates that $\mathbb{R}^3$ is a vector space.

Since they are clearly independent (why?!), the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$. This particular basis is called the **standard basis** of $\mathbb{R}^3$. (In general, the standard basis of $\mathbb{R}^n$ is given by the columns of the $n \times n$ identity matrix.) Since the basis consists of 3 vectors, the dimension of $\mathbb{R}^3$ is 3.

Example 103. Let $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$. What is $\dim V$?

Solution. The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are clearly dependent (why?!). Since, $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have that $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$. In particular, $\dim V = 2$.

[By the way, we still say that V is a subspace of $\mathbb{R}^2$ even though, as was the case here, V might be all of $\mathbb{R}^2$.]

Armin Straub
straub@southalabama.edu