Review. Linear independence, matrix inverses and such...

Example 89. Suppose that A is a 3×3 matrix, and that $A\boldsymbol{x} = \boldsymbol{0}$ has the solution $\boldsymbol{x} = [\,2 \;\; 1 \;\; 3\,]^T$. What does that tell us about the columns of A?

Solution. It tells us that $2\begin{bmatrix} \text{col 1} \\ \text{of} \\ A \end{bmatrix} + 1\begin{bmatrix} \text{col 2} \\ \text{of} \\ A \end{bmatrix} + 3\begin{bmatrix} \text{col 3} \\ \text{of} \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In particular, the columns of A are linearly dependent.

Here, and in what follows it is crucial to keep the following in mind:

$A\boldsymbol{x}$ is a linear combination of the columns of A.

Namely, $A\boldsymbol{x} = x_1\begin{bmatrix} \text{col 1} \\ \text{of} \\ A \end{bmatrix} + x_2\begin{bmatrix} \text{col 2} \\ \text{of} \\ A \end{bmatrix} + \cdots$

Some properties of matrix inverses

The first four statements of the next big theorem follow from Theorem 87.

Theorem 90. Let A be an $n \times n$ matrix. The following statements are all equivalent:

[That is, if one of these is true, then so are all the others.]

A is invertible.

$\Longleftrightarrow$ The RREF of A is I_n.

$\Longleftrightarrow$ A has n pivots. (Easy to check!)

$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

$\Longleftrightarrow$ The system $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution.

Why? Because if the solution is unique, then there are no free variables: every column contains a pivot, and so A has n pivots.

[There is nothing special about $\boldsymbol{0}$ here; you could replace $\boldsymbol{0}$ with any specific vector in $\mathbb{R}^n$.]

$\Longleftrightarrow$ The columns of A are linearly independent.

Why? This is the same as saying $A\boldsymbol{x} = \boldsymbol{0}$ only has the trivial solution.

$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

Why? This is a weaker statement than the one above, which also guarantees each system to have only one solution. On the other hand, if $A\boldsymbol{x} = \boldsymbol{b}$ always has a solution, then that means in an echelon form of A there cannot be a zero row because otherwise a well-chosen $\boldsymbol{b}$ would create an inconsistency (see Theorem 27). But that means every row contains a pivot, and so A has n pivots.

$\Longleftrightarrow$ The columns of A span all of $\mathbb{R}^n$.

Why? Because this is just different language for saying $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b} \in \mathbb{R}^n$. (To see that, recall that $A\boldsymbol{x}$ is a linear combination of the columns of A.)

Example 91. We now see at once that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Why? Because it has only one pivot.

Example 92. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution. Let us do elimination on $[\ A \ \ I_3\ ]$. If A is invertible, then the RREF will have the form $[\ I_3 \ \ B\ ]$ and $B = A^{-1}$.

$$\left[\begin{array}{cccccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R2 \to R2 + \frac{3}{2}R1}{\rightsquigarrow} \left[\begin{array}{cccccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{\substack{R1 \to \frac{1}{2}R1 \\ R2 \leftrightarrow R3}}{\rightsquigarrow} \left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$.

Example 93. Solve $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution. From the previous problem, we know that the matrix $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible. Using its inverse, we find that

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

is the unique solution.

Note how easy it is now (once we have the inverse) to solve linear systems with changing right-hand side:

Example 94. Solve $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Solution. Likewise, we have the unique solution

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

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