Review 76. Are the vectors
$$\begin{bmatrix} 1\\1\\1 \end{bmatrix}$$
, $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$, $\begin{bmatrix} -1\\1\\3 \end{bmatrix}$ linearly independent?

Solution. The vectors are linearly independent if and only if the system

Γ	1	1	-1]	$\begin{bmatrix} 0 \end{bmatrix}$
	1	2	1	x =	0
L	1	3	3		0

has only the trivial solution x = 0. By the usual steps of elimination, we find

Γ	1	1	-1	DDEE	1	0	-3]
	1	2	1		0	1	2	
L	1	3	3		0	0	0	

Since there is a free variable, the solution x = 0 is not unique. Hence, the vectors are linearly dependent.

- Note that we didn't include the zero vector as the right-hand side to our matrix. Why was that OK?!
- From the RREF, we can immediately read off a non-trivial solution: choose $x_3 = 1$ (or any other nonzero value). Then $x_1 = 3$, $x_2 = -2$. What is the corresponding dependence relation between our three vectors?!

Example 77. With the minimum amount of work, decide whether the following vectors are linearly independent.

	$\begin{bmatrix} 2 \end{bmatrix}$		-1		$\begin{bmatrix} 3 \end{bmatrix}$
(a)	0	,	1	,	7
	0		0		3

Solution. These vectors are linearly independent.

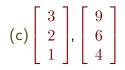
Put them as columns of a matrix, and notice that this matrix is already in echelon form...

	2		[0]	
(b)	0	,	0	
			0	

Solution. These vectors are linearly dependent.

For instance, $0\begin{bmatrix} 2\\0\\0\end{bmatrix} + 7\begin{bmatrix} 0\\0\\0\end{bmatrix} = \begin{bmatrix} 0\\0\\0\end{bmatrix}$ is a non-trivial dependence relation (the coefficients are 0 and 7).

Moral: whenever the zero vector is involved, the vectors are linearly dependent.



Solution. These vectors are linearly independent.

If they were dependent, then $x_1\begin{bmatrix} 3\\2\\1\end{bmatrix} + x_2\begin{bmatrix} 9\\6\\4\end{bmatrix} = 0$. Since $x_1 \neq 0$ (why?), $\begin{bmatrix} 3\\2\\1\end{bmatrix} = -\frac{x_2}{x_1}\begin{bmatrix} 9\\6\\4\end{bmatrix}$ so that the second vector would be a multiple of the first. But it isn't! (Judging by the first entry, the second vector would have to be 3 times the first; but that clashes with the third entry.)

Moral: two vectors are linearly dependent \iff one is a multiple of the other

$(\mathsf{d}) \left[\begin{array}{c} 1\\1 \end{array} \right], \left[\begin{array}{c} 2\\3 \end{array} \right], \left[\begin{array}{c} -1\\1 \end{array} \right]$

Moral: more than n vectors in \mathbb{R}^n are always linearly dependent.

The last observation is important enough to record it separately.

Theorem 78. More than \overline{n} vectors in \mathbb{R}^n are always linearly dependent.

Why? Put the vectors as columns of a matrix (so this matrix has n rows and more than n columns): there is only room for n pivots, but there is more than n columns. Hence, there is at least one column without pivot, and hence at least one free variable.

Matrix inverses

Throughout the discussion on matrix inverses, A is always a $n \times n$ matrix (a square matrix).

Example 79.

• $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ • $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Recall that on the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Definition 80. A matrix A is invertible if it has an **inverse**, denoted A^{-1} , satisfying

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

- If one of the conditions is true, the other is automatically true.
- If it exists, the inverse is unique. And so A^{-1} refers to a specific matrix.

Example 81. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ by the previous example.