Review 76. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ linearly independent?

Solution. The vectors are linearly independent if and only if the system

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution $\boldsymbol{x} = \mathbf{0}$. By the usual steps of elimination, we find

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a free variable, the solution $\boldsymbol{x} = \mathbf{0}$ is not unique. Hence, the vectors are linearly dependent.

- Note that we didn't include the zero vector as the right-hand side to our matrix. Why was that OK?!
- From the RREF, we can immediately read off a non-trivial solution: choose $x_3 = 1$ (or any other nonzero value). Then $x_1 = 3$, $x_2 = -2$. What is the corresponding dependence relation between our three vectors?!

Example 77. With the minimum amount of work, decide whether the following vectors are linearly independent.

(a) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$

Solution. These vectors are linearly independent.

Put them as columns of a matrix, and notice that this matrix is already in echelon form...

(b) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution. These vectors are linearly dependent.

For instance, $0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a non-trivial dependence relation (the coefficients are 0 and 7).

Moral: whenever the zero vector is involved, the vectors are linearly dependent.

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(c) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}$

Solution. These vectors are linearly independent.

If they were dependent, then $x_1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} = \mathbf{0}$. Since $x_1 \neq 0$ (why?), $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = -\frac{x_2}{x_1}\begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}$ so that the second vector would be a multiple of the first. But it isn't! (Judging by the first entry, the second vector would have to be 3 times the first; but that clashes with the third entry.)

Moral: two vectors are linearly dependent $\iff$ one is a multiple of the other

(d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Moral: more than n vectors in $\mathbb{R}^n$ are always linearly dependent.

The last observation is important enough to record it separately.

Theorem 78. More than n vectors in $\mathbb{R}^n$ are always linearly dependent.

Why? Put the vectors as columns of a matrix (so this matrix has n rows and more than n columns): there is only room for n pivots, but there is more than n columns. Hence, there is at least one column without pivot, and hence at least one free variable.

Matrix inverses

Throughout the discussion on matrix inverses, A is always a $n \times n$ matrix (a square matrix).

Example 79.

- $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  Recall that on the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Definition 80. A matrix A is invertible if it has an **inverse**, denoted A^{-1}, satisfying

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

- If one of the conditions is true, the other is automatically true.
- If it exists, the inverse is unique. And so A^{-1} refers to a specific matrix.

Example 81. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ by the previous example.

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