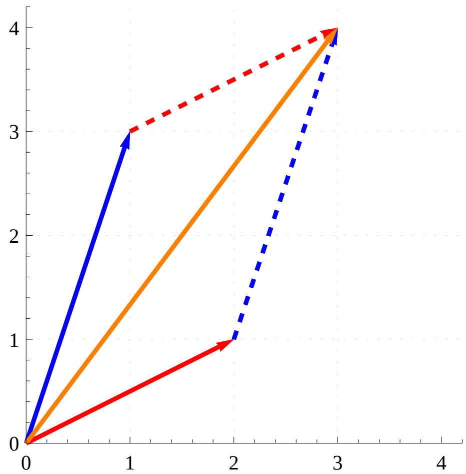
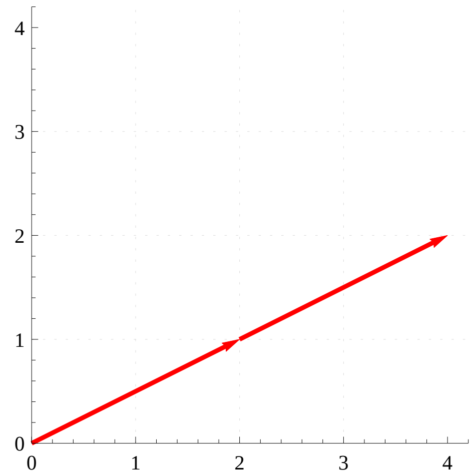
Example 33. If $\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = \begin{bmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{bmatrix}$.

Example 34. (Geometric description of $\mathbb{R}^2$) A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents the point (x_1, x_2) in the plane. Better: an arrow from the origin to (x_1, x_2)

Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, graph $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x} + \boldsymbol{y}$, $2\boldsymbol{y}$.





Example 35. Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 36. Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution. We have to find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

This is the same as:

$$\begin{array}{rrcl} 2c_1 & -c_2 & = & 1 \\ c_1 & +c_2 & = & 5 \end{array}$$

Solving, we find $c_1 = 2$ and $c_2 = 3$.

Indeed,

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Example 37. Express $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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The row and column picture

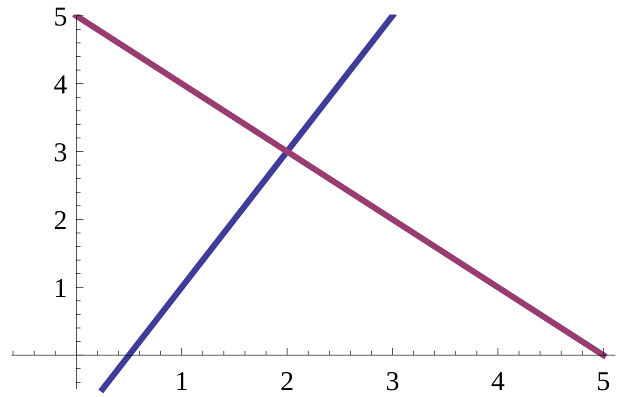
Example 38. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways. Here, there is a unique solution: $x = 2$, $y = 3$.

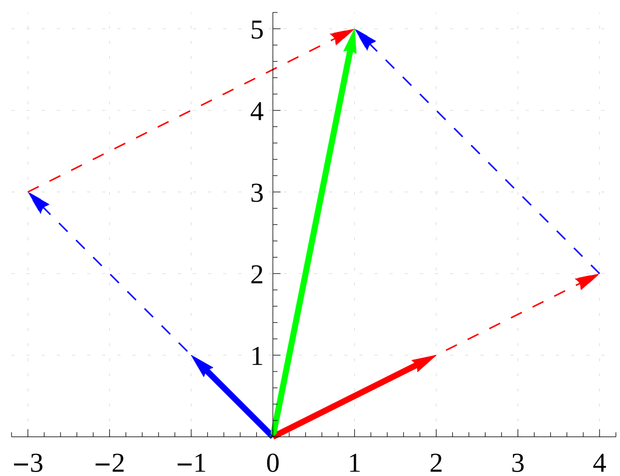
Row picture.

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.



Column picture.

- The system can be written as
$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$
- Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?
- $(2, 3)$ are the coefficients of the (only) such linear combination.



Definition 39. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all linear combinations

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m,$$

where $x_1, x_2, \ldots, x_m$ can be any real numbers. We write $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ for this set.

Example 40.

- $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ consists of all multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Geometrically, this is a line.
- $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ consists of all vectors in $\mathbb{R}^2$ (the full plane). Why?!

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