Review 24. The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

is in echelon form but not in (row-)reduced echelon form (RREF). To put it in RREF, we do

$$\overset{\substack{R_1-2R_3\Rightarrow R_1 \\ R_2-3R_3\Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

We can now read off the general solution: x_2 is a free variable, and we set $x_2 = s_1$. Then, the general solution is

$$\begin{aligned} x_1 &= -3 - 2s_1 \\ x_2 &= s_1 \\ x_3 &= -2 \\ x_4 &= 1 \end{aligned}$$

where s_1 can be any number.

Theorem 25. (Uniqueness of the reduced echelon form) Each matrix is row equivalent to one and only one reduced echelon matrix.

Determining the number of solutions of a system

Example 26. Find the general solution of the following linear system:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 1 \end{array}$$

Solution. We do Gaussian elimination to produce an echelon form:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right] \overset{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-R_1\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & -2 \end{array}\right] \overset{R_3-R_2\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & 0 & -6 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Note that the last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 4$, or $0 = 4$, which is obviously false. This means that this system cannot have a solution. It is inconsistent.

The next result says that, once we put a system in echelon form, this is the only way in which it can be inconsistent.

Theorem 27. (Existence and uniqueness theorem) A linear system in echelon form is **inconsistent** if and only if it has a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \ldots & 0 & b \end{array}\right],$$

where b is nonzero.

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If a linear system is consistent, then the solutions consist of either

- a unique solution (when there are no free variables) or
- infinitely many solutions (when there is at least one free variable).

Example 28. For what values of h will the following system be consistent?

$$\begin{array}{rrcl} 3x_1 & -9x_2 & = & 4 \\ -2x_1 & +6x_2 & = & h \end{array}$$

Solution. We perform row reduction to find an echelon form:

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \overset{R_2+\frac{2}{3}R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{rr|r} 3 & -9 & 4 \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

Whether or not the system is consistent is determined by whether or not $h+\frac{8}{3}=0$.
The system is consistent if and only if $h=-\frac{8}{3}$.

(In the [single] case in which it is consistent, it has infinitely many solutions [because x_2 is a free variable].)

Adding and scaling vectors

Example 29. We have already encountered **matrices** such as

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 2 \\ 3 & 2 & -2 & 0 \end{bmatrix}.$$

Each column is what we call a **(column) vector**.
In this example, each column vector has 3 entries and so lies in $\mathbb{R}^3$.

Example 30. A fundamental property of vectors is that vectors of the same kind can be **added** and **scaled**.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}, \qquad 7 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{bmatrix}.$$

Adding and scaling vectors, the most general thing we can do is:

Definition 31. Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_m$, the vector

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m$$

is a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

The scalars $c_1, \ldots, c_m$ are the **coefficients** or **weights**.

Example 32. Linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ include:

- $3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 7\boldsymbol{v}_3$,
- $\boldsymbol{v}_2 + \boldsymbol{v}_3$,
- $\frac{1}{3}\boldsymbol{v}_2$,
- $\mathbf{0}$ (the zero vector).

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