Matrix notation

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{rr|r} 2 & -4 & -2 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

Example 11. What is the augmented matrix for the following linear system?

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 5 \end{array}$$

Example 12. Let us solve the system in matrix notation.

$$\left[\begin{array}{rr|r} 2 & -4 & -2 \\ -1 & 3 & 3 \end{array}\right] \qquad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$\underset{\rightsquigarrow}{R_2+\tfrac{1}{2}R_1 \Rightarrow R_2} \left[\begin{array}{rr|r} 2 & -4 & -2 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ & & x_2 & = & 2 \end{array}$$

Hence, $x_2 = 2$ and, by back-substitution, we find (from $2x_1 - 4 \cdot 2 = -2$) that $x_1 = 3$.

Alternatively, instead of back-substitution, we can also continue with row operations:

$$\underset{\rightsquigarrow}{R_1+4R_2 \Rightarrow R_1} \left[\begin{array}{rr|r} 2 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcr} 2x_1 & & = 6 \\ & x_2 & = 2 \end{array}$$

$$\underset{\rightsquigarrow}{\tfrac{1}{2}R_1 \Rightarrow R_1} \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcr} x_1 & & = 3 \\ & x_2 & = 2 \end{array}$$

Definition 13. An **elementary row operation** is one of the following:

- **(replacement)** Add one row to a multiple of another row.
- **(interchange)** Interchange two rows.
- **(scaling)** Multiply all entries in a row by a nonzero constant.

Definition 14. Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 15. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

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Example 16. Let us solve the following system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & - & 4x_2 & - & 5x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & -4 & -5 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & - & 4x_2 & - & 5x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{c} R_2-3R_1\Rightarrow R_2 \\ R_3+4R_1\Rightarrow R_3 \\ \rightsquigarrow \end{array} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

$$\begin{array}{c} R_3+\frac{3}{2}R_2\Rightarrow R_3 \\ \rightsquigarrow \end{array} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

We could stop now and do back-substitution (do it!!), or we can continue:

$$\begin{array}{c} R_1-R_3\Rightarrow R_1 \\ R_2+8R_3\Rightarrow R_2 \\ \rightsquigarrow \end{array} \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & 2x_2 & & & = & 32 \\ & & & & x_3 & = & 3 \end{array}$$

$$\begin{array}{c} R_1+R_2\Rightarrow R_1 \\ \frac{1}{2}R_2\Rightarrow R_2 \\ \rightsquigarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array}$$

We find the solution $(x_1, x_2, x_3) = (29, 16, 3)$. (Do you see how this system is related to the one from last class?)

Note that we are aiming for a stair-case shape in our approach. More precisely:

Definition 17. A matrix is in **echelon form** (or **row echelon form**) if:

(a) Each leading entry (i.e. leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.

(b) All zero rows are at the bottom of the matrix.

Example 18. Here is a representative matrix in echelon form.

$$\left[\begin{array}{ccccccccccc} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

($*$ stands for any value, and $\blacksquare$ for any nonzero value.)

Definition 19.

- A leading entry in an echelon form is called a **pivot**.
- A column containing a pivot is called a **pivot column**.

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