# Homework #9

**MATH 237 — Linear Algebra I**
**due Thursday, Dec 3, in class**

***Please print your name:***

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**Problem 1.** For each matrix $A$, find the eigenvalues of $A$ as well as a basis for the corresponding eigenspaces.

(a) $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution.**

(a) By expanding by the third row, we find that the characteristic polynomial is

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda).$$

Hence, the eigenvalues are $\lambda=-1$, $\lambda=2$ and $\lambda=4$.

- For $\lambda=-1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$.
- For $\lambda=2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -3 & 0 & 1 \\ -3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.
- For $\lambda=4$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) By expanding by the third row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)((2-\lambda)^2-1) = (1-\lambda)(\lambda-3)(\lambda-1).$$

Hence, the eigenvalues are $\lambda=1$ (with multiplicity 2) and $\lambda=3$.

- For $\lambda=3$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
- For $\lambda=1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. □

**Problem 2.** The processors of a supercomputer are inspected weekly in order to determine their condition. The condition of a processor can either be perfect, good, reasonable or bad.

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A perfect processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good, and with probability 0.1 it is reasonable. A processor in good conditions is still good after one week with probability 0.6, reasonable with probability 0.2, and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The reparation takes one week, after which the processor is again in perfect condition.

In the steady state, what is the percentage of processors in perfect condition?

**Solution.** We consider four states: perfect, good, reasonable, bad

The transition matrix is:

$$\begin{bmatrix} 0.7 & 0 & 0 & 1 \\ 0.2 & 0.6 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0 \\ 0 & 0.2 & 0.5 & 0 \end{bmatrix}$$

The steady state is the eigenvector corresponding to the eigenvalue 1 (with the extra condition that summation of the entries of the vector should be 1; since the states are percentages). Hence, we determine null($A$) with

$$A = \begin{bmatrix} 0.7-1 & 0 & 0 & 1 \\ 0.2 & 0.6-1 & 0 & 0 \\ 0.1 & 0.2 & 0.5-1 & 0 \\ 0 & 0.2 & 0.5 & 0-1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0 & 0 & 1 \\ 0.2 & -0.4 & 0 & 0 \\ 0.1 & 0.2 & -0.5 & 0 \\ 0 & 0.2 & 0.5 & -1 \end{bmatrix}.$$

Here is some observations that save us a little bit of time (but it is perfectly fine if you just do the usual elimination with $A$). Note that $R_1 + R_2 + R_3 + R_4 = [\,0\ \ 0\ \ 0\ \ 0\,]$. (This is a consequence of the fact that we started with a stochastic matrix; can you see that?)

$$\begin{bmatrix} -0.3 & 0 & 0 & 1 \\ 0.2 & -0.4 & 0 & 0 \\ 0.1 & 0.2 & -0.5 & 0 \\ 0 & 0.2 & 0.5 & -1 \end{bmatrix} \overset{R_1 \Rightarrow R_1 + R_2 + R_3 + R_4}{\rightsquigarrow} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & -0.4 & 0 & 0 \\ 0.1 & 0.2 & -0.5 & 0 \\ 0 & 0.2 & 0.5 & -1 \end{bmatrix}$$

This matrix is in some sort of echelon form (if we reorder the equations and variables, then it is in a "real" echelon form), and so we can solve the homogeneous system by back-substitution with $x_1$ as our free variable.

We find that the eigenspace of the eigenvalue 1 is spanned by $\begin{bmatrix} 1 \\ 1/2 \\ 2/5 \\ 3/10 \end{bmatrix}$.

Since $1 + \frac{1}{2} + \frac{2}{5} + \frac{3}{10} = \frac{22}{10}$, the steady state is $\frac{10}{22}\begin{bmatrix} 1 \\ 1/2 \\ 2/5 \\ 3/10 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 10 \\ 5 \\ 4 \\ 3 \end{bmatrix}$.

In particular, in the steady state $\frac{10}{22} \approx 45.5\%$ of processors are in perfect condition. $\square$

**Problem 3.** Consider $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

(a) What is the geometric interpretation of the linear map $\boldsymbol{x} \mapsto A\boldsymbol{x}$, which maps $\mathbb{R}^2$ to $\mathbb{R}^2$?

[It helps to make a sketch of where the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are being sent. If still stuck, look again at Example 145.]

(b) What is the geometric interpretation of the linear map $\boldsymbol{x} \mapsto A^2\boldsymbol{x}$? (No computation here.)

(c) Compute $A^2$ by multiplying $A$ with itself.

[From your interpretations it follows that $A^2 = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$. By comparing the bottom left entries of this with your computation of $A^2$, you can conclude that $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$. You proved a trig identity!]

**Solution.**

(a) This is rotation about the origin by angle $\theta$ (in radians and counter-clockwise).

(b) This is rotation about the origin by angle $2\theta$.

(c) $A^2 = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$

Comparing this with $A^2 = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$, we conclude that $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (which can be simplified to $\cos(2\theta) = 2\cos^2(\theta) - 1$). These are trig identities that you have surely seen before. Now, you have a simple way of deducing them!

[If you enjoyed this: can you generalize our approach to find a trig identity for $\cos(\theta + \rho)$ and $\sin(\theta + \rho)$?]

$\square$