Homework #6

MATH 237 — Linear Algebra I
due Thursday, Oct 22, in class

Please print your name:

Problem 1. Let $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$.

(a) Find a basis for col(A). What is the dimension of col(A)?

(b) Is the vector $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in col(A)?

Hint: Note that this is the same question as: "Does $A\boldsymbol{x} = \boldsymbol{u}$ have a solution?"
However, save yourself time and observe that if, say, $\boldsymbol{v}_1, \boldsymbol{v}_2$ form a basis for col(A), then you only need to determine whether the simpler system $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{u}$ has a solution (because we got rid of free variables, this system either has a unique solution or none at all).

(c) Find a basis for col(A^T). What is the dimension of col(A^T)?

(d) Is the vector $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$ in col(A^T)? Is $\boldsymbol{w}$ in col(A)? Conclude that col(A) $\neq$ col(A^T) (but both spaces have the same dimension).

(e) If possible, write the vector $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \\ 13 \\ 12 \end{bmatrix}$ as a linear combination of your basis of col(A).

Solution.

(a) We compute an echelon form of A:

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \overset{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-3R_1\Rightarrow R_3 \\ R_4-4R_1\Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_3+2R_2\Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are the first and third. Hence, a basis for col(A) is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

In particular, the dimension of col(A) is 2.

(b) Since $\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$, we need to decide whether $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \boldsymbol{u}$ has a solution.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 3 & 2 & 0 \\ 4 & 0 & 0 \end{array}\right] \overset{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-3R_1\Rightarrow R_3 \\ R_4-4R_1\Rightarrow R_4}}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & -4 \end{array}\right]$$

We stop here because we can already tell that this system is inconsistent. Hence, $\boldsymbol{u}$ is not in col(A).

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(c) We compute an echelon form of A^T:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & -1 & 2 & 0 \\ 4 & 3 & 22 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2-2R_1\Rightarrow R_2 \\ R_4-4R_1\Rightarrow R_4}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix} \underset{\rightsquigarrow}{\substack{\text{swap zero} \\ \text{row to bottom}}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{R_3-5R_2\Rightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are the first and second. Hence, a basis for $\mathrm{col}(A^T)$ is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \\ 3 \end{bmatrix}$.

In particular, the dimension of $\mathrm{col}(A^T)$ is 2.

(d) The vector $\boldsymbol{w}$ is obviously in $\mathrm{col}(A^T)$, because it is the first column of A^T.

To see whether $\boldsymbol{w}$ is in $\mathrm{col}(A)$, we proceed as we did for $\boldsymbol{u}$ (note that, for the first step, the row operations are the same; only the right-hand side changes):

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 2 & 0 \\ 4 & 0 & 4 \end{array}\right] \underset{\rightsquigarrow}{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-3R_1\Rightarrow R_3 \\ R_4-4R_1\Rightarrow R_4}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{array}\right] \underset{\rightsquigarrow}{R_3+2R_2\Rightarrow R_3} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

This system is inconsistent. Hence, $\boldsymbol{w}$ is not in $\mathrm{col}(A)$.

Because $\boldsymbol{w}$ is in $\mathrm{col}(A^T)$ but not in $\mathrm{col}(A)$, we see that these are two different spaces. However, they both are subspaces of $\mathbb{R}^4$ and have dimension 2.

(e) We proceed as we did for the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 3 & 2 & 13 \\ 4 & 0 & 12 \end{array}\right] \underset{\rightsquigarrow}{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-3R_1\Rightarrow R_3 \\ R_4-4R_1\Rightarrow R_4}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right] \underset{\rightsquigarrow}{R_3+2R_2\Rightarrow R_3} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This system has solution $x_1 = 3$ and $x_2 = 2$, which implies that $\boldsymbol{a} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$. □

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