EXERCISES

For Exercises 1–6, determine if the given vectors are linearly independent.

1. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 6 \\ -15 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ -10 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \\ -13 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -8 \\ 2 \\ -19 \end{bmatrix}$

5. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$

6. $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 5 \\ -5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 7–12, determine if the columns of the given matrix are linearly independent.

7. $\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & -12 \\ 2 & 6 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 5 & -7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -5 \\ -1 & 2 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 1 & 0 \\ 5 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -4 & -7 & 1 \\ 0 & 0 & 3 \\ 5 & -1 & 1 \\ 8 & 2 & -4 \end{bmatrix}$

In Exercises 13–18, a matrix A is given. Determine if the homogeneous system $A\mathbf{x} = \mathbf{0}$ (where $\mathbf{x}$ and $\mathbf{0}$ have the appropriate number of components) has any nontrivial solutions.

13. $A = \begin{bmatrix} -3 & 5 \\ 4 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 12 & 10 \\ 6 & 5 \end{bmatrix}$

15. $A = \begin{bmatrix} 8 & 1 \\ 0 & -1 \\ -3 & 2 \end{bmatrix}$

16. $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ 5 & -4 & -3 \end{bmatrix}$

17. $A = \begin{bmatrix} -1 & 3 & 1 \\ 4 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 2 \\ -5 & 3 & -9 \\ 3 & 0 & 9 \end{bmatrix}$

In Exercises 19–24, determine by inspection (that is, with only minimal calculations) if the given vectors form a linearly dependent or linearly independent set. Justify your answer.

19. $\mathbf{u} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

20. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

21. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

23. $\mathbf{u} = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -7 \\ 1 \\ 12 \end{bmatrix}$

24. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 25–28, determine if one of the given vectors is in the span of the other vectors. (HINT: Check to see if the vectors are linearly dependent, and then appeal to Theorem 2.14.)

25. $\mathbf{u} = \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$

26. $\mathbf{u} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

27. $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -5 \\ 7 \\ -7 \end{bmatrix}$

28. $\mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ 8 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

For each matrix A given in Exercises 29–32, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$. (HINT: the Big Theorem is helpful here.)

29. $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ -3 & 4 & 5 \end{bmatrix}$

30. $A = \begin{bmatrix} 3 & 4 & 7 \\ 7 & -1 & 6 \\ -2 & 0 & 2 \end{bmatrix}$

31. $A = \begin{bmatrix} 3 & -2 & 1 \\ -4 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & 7 \end{bmatrix}$

FIND AN EXAMPLE For Exercises 33–38, find an example that meets the given specifications.

33. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^4$.

34. Three linearly independent vectors in $\mathbf{R}^5$.

35. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^2$ that do not span $\mathbf{R}^2$.

36. Three distinct nonzero vectors in $\mathbf{R}^2$ such that any pair is linearly independent.

37. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^3$ such that each vector is in the span of the other two vectors.

38. Four vectors in $\mathbf{R}^3$ such that no vector is a nontrivial linear combination of the other three. (Explain why this does not contradict Theorem 2.14.)

TRUE OR FALSE For Exercises 39–52, determine if the statement is true or false, and justify your answer.

39. If a set of vectors in $\mathbf{R}^n$ is linearly dependent, then the set must span $\mathbf{R}^n$.

40. If $m > n$, then a set of m vectors in $\mathbf{R}^n$ is linearly dependent.

41. If A is a matrix with more rows than columns, then the columns of A are linearly independent.

42. If A is a matrix with more columns than rows, then the columns of A are linearly independent.

43. If A is a matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

44. If A is a matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$.

45. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

46. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

47. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

48. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

49. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

50. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

51. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

52. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

53. Which of the following sets of vectors in $\mathbf{R}^3$ could possibly be linearly independent? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

54. Which of the following sets of vectors in $\mathbf{R}^3$ could possibly be linearly independent *and* span $\mathbf{R}^3$? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

55. Prove that if c_1, c_2, and c_3 are nonzero scalars and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors, then so is $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, c_3\mathbf{u}_3\}$.

56. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors, then so are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

57. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors, then so is $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$.

58. Prove that if $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly independent, then any nonempty subset of U is also linearly independent.

59. Prove that if a set of vectors is linearly dependent, then adding additional vectors to the set will create a new set that is still linearly dependent.

60. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set, then $\mathbf{w}$ is in span$\{\mathbf{u}, \mathbf{v}\}$.

61. Prove that two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if $\mathbf{u} = c\mathbf{v}$ for some scalar c.

62. Let A be an $n \times m$ matrix that is in echelon form. Prove that the nonzero rows of A, when considered as vectors in $\mathbf{R}^m$, are a linearly independent set.

63. Prove part (b) of Theorem 2.16.

64. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a linearly dependent set of nonzero vectors. Prove that some vector in the set can be written as a linear combination of a linearly independent subset of the remaining vectors, with the set of coefficients all nonzero and unique for the given subset. (HINT: Start with Theorem 2.14.)

In Exercises 65–66, suppose that the given vectors are direction vectors for a model of the VecMobile III (discussed in Section 2.2). Determine if there is any redundancy in the vectors and if it is possible to reach every point in $\mathbf{R}^3$.

65. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$

66. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix}$

C In Exercises 67–70, determine if the given vectors form a linearly dependent or linearly independent set.

67. $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix}$

68. $\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix}$

69. $\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \\ -3 \end{bmatrix}$

70. $\begin{bmatrix} 3 \\ 5 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 2 \\ 6 \end{bmatrix}$

C In Exercises 71–72, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$.

71. $A = \begin{bmatrix} 1 & -2 & 4 \\ 5 & -3 & -1 \\ -3 & -7 & -9 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

72. $A = \begin{bmatrix} 3 & -2 & 5 \\ 2 & 0 & -4 \\ -2 & 7 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

C In Exercises 73–74, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^4$.

73. $A = \begin{bmatrix} 2 & 5 & -3 & 6 \\ -1 & 0 & 1 & -1 \\ 5 & 2 & -3 & 9 \\ 3 & -4 & 6 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

74. $A = \begin{bmatrix} 5 & 1 & 0 & 8 \\ -2 & 4 & 3 & 11 \\ -3 & 8 & 2 & 5 \\ 0 & 3 & -1 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$