Review

Let A be $n \times n$ with independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

Then A can be **diagonalized** as $A = PDP^{-1}$.

Example 1. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Solution.

Example 2. Suppose $A = PDP^{-1}$. Then, what is A^n?

Solution.

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Example 3. The **differential equation** $y' = ay$ with **initial condition** $y(0) = C$ is solved $y(t) = Ce^{at}$. (This solution is unique.)

Why?

Example 4. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcl} y_1' &=& 2y_1 \\ y_2' &=& -y_1 + 3y_2 + y_3 \\ y_3' &=& -y_1 + y_2 + 3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form:

Key idea: to solve $\boldsymbol{y}' = A\boldsymbol{y}$, introduce e^{At}

Definition 5. Let A be $n \times n$. The **matrix exponential** is

$$e^A =$$

It shares many properties of the usual exponential:

- e^A is invertible and $(e^A)^{-1} =$
- $e^A e^B = e^{A+B} = e^B e^A$ if $AB = BA$
- $\frac{\mathrm{d}}{\mathrm{d}t} e^{At} =$
- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} =$

Example 6. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then:

$$e^A =$$

$$e^{At} =$$

Clearly, this works to obtain e^D for any diagonal matrix D.

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Theorem 7. Suppose $A = PDP^{-1}$. Then, $e^A =$

Why?

Example 8. (continued) We wish to solve:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} =$

$$A \;= PDP^{-1} \quad \text{with} \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$$

$$e^{At}$$

Check (optional) that $\boldsymbol{y} = \begin{bmatrix} \quad \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' =$$

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Example 9. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution.