Application: Fourier series

Review. Given an orthogonal basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$, we express a vector $\boldsymbol{x}$ as

$$\boldsymbol{x} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \ldots, \qquad c_i =$$

A **Fourier series** of a function $f(x)$ is an infinite expansion:

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

- We are working in the infinite dimensional vector space of functions.

  More precisely, we are working with (say, continuous) functions that are periodic with period 2π.

- The functions

  $$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$$

  are a basis of this space. In fact, an **orthogonal basis**!

  That's the reason for the success of Fourier series.

What is the inner product on the space of functions?

- Vectors: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$
- Functions: $\langle f, g \rangle =$

  Why these limits?

Example 1. Show that $\cos(x)$ and $\sin(x)$ are orthogonal.

Solution.

More generally, $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$ are all orthogonal to each other.

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Example 2. What is the norm of $\cos(x)$?

Solution.

Example 3. How do we find a_1?

Or: how much cosine is in a function $f(x)$?

Solution.

$f(x)$ has the Fourier series

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

where

$$a_k = \frac{\langle f(x), \cos(kx)\rangle}{\langle \cos(kx), \cos(kx)\rangle} =$$

$$b_k = \frac{\langle f(x), \sin(kx)\rangle}{\langle \sin(kx), \sin(kx)\rangle} =$$

$$a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} =$$

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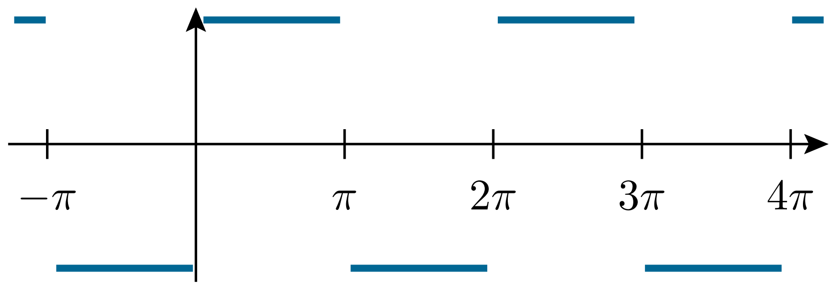
Example 4. Find the Fourier series of the 2π-periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0), \\ +1, & \text{for } x \in (0, \pi). \end{cases}$$



Solution.

Note. We just observed the following general principle: an odd function is orthogonal to ...

$f(x)$ is odd and the cosines are even functions, so ...

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Example 5. Consider the space of 1-periodic functions.

- What does a Fourier series for a 1-periodic $f(x)$ look like?
- What should be our inner product for Fourier series?
- How are the Fourier coefficients computed?

Solution.

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