Vector spaces and subspaces

We have already encountered **vectors** in $\mathbb{R}^n$. Now, we discuss the general concept of vectors.

In place of the space $\mathbb{R}^n$, we think of general **vector spaces**.

Definition 1. A **vector space** is a nonempty set V of elements, called **vectors**, which may be added and scaled (multiplied with real numbers).

The two operations of addition and scalar multiplication must satisfy the following *axioms* for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in V, and all scalars c, d.

(a) $\boldsymbol{u} + \boldsymbol{v}$ is in V

(b) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

(c) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

(d) there is a vector (called the **zero vector**) $\boldsymbol{0}$ in V such that $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$ for all $\boldsymbol{u}$ in V

(e) there is a vector $-\boldsymbol{u}$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$

(f) $c\boldsymbol{u}$ is in V

(g) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$

(h) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$

(i) $(cd)\boldsymbol{u} = c(d\boldsymbol{u})$

(j) $1\boldsymbol{u} = \boldsymbol{u}$

tl;dr — A **vector space** is a collection of vectors which can be added and scaled; subject to the usual rules you would hope for.

namely: associativity, commutativity, distributivity

Armin Straub
astraub@illinois.edu

Example 2. Convince yourself that $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$ is a vector space.

Solution. In this context, the zero vector is $\mathbf{0} =$

Example 3. Let $\mathbb{P}_n$ be the set of all polynomials of degree at most $n \geqslant 0$. Is $\mathbb{P}_n$ a vector space?

Solution.

Example 4. Let V be the set of all polynomials of degree exactly 3. Is V a vector space?

Solution.

Example 5. Let V be the set of all functions $f: \mathbb{R} \to \mathbb{R}$. Is V a vector space?

Solution.

Subspaces

Definition 6. A subset W of a vector space V is a **subspace** if W is itself a vector space.

Armin Straub
astraub@illinois.edu

Since the rules like associativity, commutativity and distributivity still hold, we only need to check the following:

$W \subseteq V$ is a subspace of V if

- W contains the zero vector $\mathbf{0}$,
- W is closed under addition, (i.e. if $\boldsymbol{u}, \boldsymbol{v} \in W$ then $\boldsymbol{u} + \boldsymbol{v} \in W$)
- W is closed under scaling. (i.e. if $\boldsymbol{u} \in W$ and $c \in \mathbb{R}$ then $c\boldsymbol{u} \in W$)

Example 7. Is $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

Solution.

Example 8. Is $W = \left\{\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \text{ in } \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

Solution.

Armin Straub
astraub@illinois.edu

Example 9. Is $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^2$?

Solution.

Example 10. Is $W = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

Solution.

Spans of vectors are subspaces

Review. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all their linear combinations. We denote it by $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

In other words, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all vectors of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m,$$

where $c_1, c_2, \ldots, c_m$ are scalars.

Theorem 11. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are in a vector space V, then $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is a subspace of V.

Example 12. Is $W = \left\{ \begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

Solution.

Example 13. Is $W = \left\{ \begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $M_{2\times 2}$, the space of 2×2 matrices?

Solution.

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Practice problems

Example 14. Are the following vector spaces?

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + 3b = 0, 2a - c = 1 \right\}$

(b) $W_2 = \left\{ \begin{bmatrix} a + c \\ -2b \\ b + 3c \\ c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

(c) $W_2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geqslant 0 \right\}$