Matrix operations

Basic notation

We will use the following notations for an $m \times n$ matrix A (m rows, n columns).

- In terms of the columns of A:

$$A = [\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \cdots \;\; \boldsymbol{a}_n \,] = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}$$

- In terms of the entries of A:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \qquad a_{i,j} = \begin{array}{c}\text{entry in} \\ i\text{-th row,} \\ j\text{-th column}\end{array}$$

Matrices, just like vectors, are added and scaled componentwise.

Example 1.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$

(b) $7 \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$

Matrix times vector

Recall that $(x_1, x_2, \ldots, x_n)$ solves the linear system with augmented matrix

$$[\, A \;\; \boldsymbol{b} \,] = \begin{bmatrix} | & | & & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \\ | & | & & | & | \end{bmatrix}$$

if and only if

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \ldots + x_n \boldsymbol{a}_n = \boldsymbol{b}.$$

It is therefore natural to define the **product of matrix times vector** as

$$A\boldsymbol{x} = x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \ldots + x_n \boldsymbol{a}_n, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

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The product of a matrix A with a vector $\boldsymbol{x}$ is a linear combination of the columns of A with weights given by the entries of $\boldsymbol{x}$.

Example 2.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

(b) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

This illustrates that linear systems can be simply expressed as $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{array}{rrcl} 2x_1 & +3x_2 & = & b_1 \\ 3x_1 & +x_2 & = & b_2 \end{array} \quad \Longleftrightarrow \quad \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Example 3. Suppose A is $m \times n$ and $\boldsymbol{x}$ is in $\mathbb{R}^p$. Under which condition does $A\boldsymbol{x}$ make sense?

Matrix times matrix

The **product of matrix times matrix** is given by

$$AB = [\ A\boldsymbol{b}_1 \ \ A\boldsymbol{b}_2 \ \ \cdots \ \ A\boldsymbol{b}_p\ ], \qquad B = [\ \boldsymbol{b}_1 \ \ \boldsymbol{b}_2 \ \ \cdots \ \ \boldsymbol{b}_p\ ].$$

Example 4.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$

because $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$

Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B.

Remark 5. The definition of the matrix product is inevitable from the multiplication of matrix times vector and the fact that we want AB to be defined such that $(AB)\boldsymbol{x} =$

$A(B\boldsymbol{x})$.

$$\begin{aligned} A(B\boldsymbol{x}) &= A(x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + \cdots) \\ &= x_1A\boldsymbol{b}_1 + x_2A\boldsymbol{b}_2 + \cdots \\ &= (AB)\boldsymbol{x} \quad \text{if the columns of } AB \text{ are } A\boldsymbol{b}_1, A\boldsymbol{b}_2, \ldots \end{aligned}$$

Example 6. Suppose A is $m \times n$ and B is $p \times q$.

(a) Under which condition does AB make sense?

(b) What are the dimensions of AB in that case?

Basic properties

Example 7.

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$

This is the 2×2 **identity matrix**.

Theorem 8. Let A, B, C be matrices of appropriate size. Then:

- $A(BC) = (AB)C$ associative
- $A(B + C) = AB + AC$ left-distributive
- $(A + B)C = AC + BC$ right-distributive

Example 9. However, matrix multiplication is not commutative!

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$

Example 10. Also, a product can be zero even though none of the factors is:

$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} =$

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Two more ways to look at matrix multiplication

Example 11. What is the entry $(AB)_{i,j}$ at row i and column j?

The j-th column of AB is $A\cdot(\text{col } j \text{ of } B)$.

Row i of that is $(\text{row } i \text{ of } A)\cdot(\text{col } j \text{ of } B)$. In other words:

$$(AB)_{i,j} = (\text{row } i \text{ of } A)\cdot(\text{col } j \text{ of } B)$$

Use this **row-column rule** to compute:

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} =$$

Observe the symmetry between rows and columns in this rule!

It follows that the interpretation

> "Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B."

has the counterpart

> "Each row of AB is a linear combination of the rows of B with weights given by the corresponding row of A."

Transpose of a matrix

Definition 12. The **transpose** A^T of a matrix A is the matrix whose columns are formed from the corresponding rows of A. rows $\leftrightarrow$ columns

Example 13.

(a) $\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}^T =$

(b) $[\, x_1 \;\; x_2 \;\; x_3 \,]^T =$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}^T =$

A matrix A is called **symmetric** if $A = A^T$.

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Example 14. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

Compute:

(a) $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} =$

(b) $(AB)^T = \begin{bmatrix} & \\ & \end{bmatrix}$

(c) $B^T A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} =$

(d) $A^T B^T$

What's that fishy smell?

Theorem 15. Let A, B be matrices of appropriate size. Then:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Example 16. Deduce that $(ABC)^T = C^T B^T A^T$.

Questions to check our understanding

- True or false?
  - AB has as many columns as B.
  - AB has as many rows as B.

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