Math 415 - Midterm 3

Thursday, November 20, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. ***[1 point]*** Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Mahmood, Michael, Nathan, Pouyan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	5	MC	$\sum$
/1	/7	/8	/8	/8	/8	/20	/60

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of 10 pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. ***[7 points]*** Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

Problem 2. *[8 points]* Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the eigenvalues of A, as well as a basis for the corresponding eigenspaces.

Problem 3. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(a) ***[6 points]*** Find a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$.

(b) ***[2 points]*** Find the least squares line for the data points $(-1, 1)$, $(0, -1)$, $(1, 1)$.

Problem 4. Find the determinant of the following matrices. Show all steps of your calculations.

(a) ***[4 points]***

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(b) ***[4 points]***

$$\begin{bmatrix} 2 & -1 & 3 & 7 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

[*Hint:* For this second matrix, begin your calculation with a row operation to save time.]

Problem 5. Let $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

(a) ***[6 points]*** Find the projection matrix corresponding to orthogonal projection onto W.

(b) ***[2 points]*** What is the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ onto W?

MULTIPLE CHOICE

(10 questions, 2 points each)

Instructions for multiple choice questions

- No reason needs to be given. There is always exactly one correct answer.
- Enter your answer on the scantron sheet that is included with your exam.
  In addition, on your exam paper, circle the choices you made on the scantron sheet.
- Use a **number 2 pencil** to shade the bubbles completely and darkly.
- Do **NOT** cross out your mistakes, but rather erase them thoroughly before entering another answer.
- Before beginning, please code in your name, UIN, and netid in the appropriate places. In the **'Section' field on the scantron**, please enter

  000 if Armin Straub is your instructor,

  001 if Philipp Hieronymi is your instructor.

MC 1. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, and let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Suppose that $\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$ is in W and $\boldsymbol{b}$ is orthogonal to W. Then:

(a) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(c) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

(d) none of the above

MC 2. Let Q be an orthogonal matrix. What is the best you can say about $\det(Q)$?

(a) $\det Q = 1$

(b) $\det Q = \pm 1$

(c) $\det Q \neq 0$

(d) $\det Q > 0$

MC 3. If A and B are 2×2 matrices with $\det(A) = 2$ and $\det(B) = -1$. What is the determinant of $C = -2ABA^T$?

(a) 4
(b) -8
(c) 8
(d) -16
(e) 16

MC 4. Let A, B be two $n \times n$-matrices. Consider the following two statements:

(S1) If $\det(A) = 0$, then two rows or two columns of A are the same, or a row or a column of A is zero.

(S2) If two row interchanges on A are made in succession to get a matrix B, then $\det(A) = \det(B)$.

Then:

(a) Statement S1 and Statement S2 are correct.
(b) Only Statement S1 is correct.
(c) Only Statement S2 is correct.
(d) Neither Statement S1 nor Statement S2 is correct.

MC 5. Which of the following choices for a makes $\begin{bmatrix} 3 & a & 0 \\ 3a & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ invertible?

(a) any real number except $-\sqrt{2}$ and $\sqrt{2}$
(b) any real number except -2 and 2
(c) 6
(d) any real number except -2 and 6

MC 6. Consider the following two statements:

(T1) If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is an orthogonal basis for W, then multiplying $\boldsymbol{v_3}$ by a scalar c gives a new orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, c\boldsymbol{v_3}\}$.

(T2) The Gram–Schmidt process produces from a linearly independent set $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ an orthonormal set $\{\boldsymbol{q_1}, \dots, \boldsymbol{q_n}\}$ with the property that for each $k \leq n$ the vectors $\{\boldsymbol{q_1}, \dots, \boldsymbol{q_k}\}$ span the same subspace as $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_k}\}$.

Then:

(a) Statement T1 and Statement T2 are correct.
(b) Only Statement T1 is correct.
(c) Only Statement T2 is correct.
(d) Neither Statement T1 nor Statement T2 is correct.

MC 7. Consider the vector space V of all continuous functions $\mathbb{R} \to \mathbb{R}$, which are periodic with period 8. What is a natural inner product on V?

(a) $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t$

(b) $\langle f, g\rangle = \int_0^{8} f(t)g(t)\mathrm{d}t$

(c) $\langle f, g\rangle = \int_0^{8} (f(t) - g(t))^2\mathrm{d}t$

(d) $\langle f, g\rangle = 8f(t)g(t)$

(e) $\langle f, g\rangle = f(1)g(1) + \ldots + f(n)g(n)$

MC 8. Consider the space $\mathbb{P}^3$ of polynomials of degree up to 3, together with the inner product

$$\langle p(t), q(t)\rangle = \int_0^1 p(t)q(t)\mathrm{d}t.$$

What is the orthogonal projection of the polynomial t onto $\mathrm{span}\{t^2\}$?

(a) $\dfrac{\int_0^1 s^3\mathrm{d}s}{\int_0^1 s^4\mathrm{d}s}\, t^2$

(b) $\dfrac{\int_0^1 s^3\mathrm{d}s}{\int_0^1 s^4\mathrm{d}s}\, t$

(c) 0

(d) t

(e) t^2

MC 9. Let A be an $n \times n$ matrix. Consider the following two statements:

(U1) The matrix $7A$ has the same eigenvectors as A.

(U2) The matrix $7A$ has the same eigenvalues as A.

Then:

(a) Statement U1 and Statement U2 are correct.

(b) Only Statement U1 is correct.

(c) Only Statement U2 is correct.

(d) Neither Statement U1 nor Statement U2 is correct.

MC 10. Let $W = \mathrm{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto W, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto W. Then:

(a) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(b) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(c) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(e) $\boldsymbol{w}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(Scratch paper)