Math 415 - Midterm 3

Thursday, November 20, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. ***[1 point]*** Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Mahmood, Michael, Nathan, Pouyan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	5	6	Shorts	$\sum$
/1	/?	/?	/?	/?	/?	/?	/?	/?

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$. Find a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 2. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find an orthonormal basis for W.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ onto W?

(c) Write $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ as the sum of a vector in W and a vector in $W^\perp$.

(d) Find the projection matrix corresponding to orthogonal projection onto W.

Problem 3. Find the QR decomposition of $A = \begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$.

Problem 4. Find the least squares line for the data points $(1,1)$, $(2,1)$, $(3,4)$, $(4,4)$.

Problem 5.

(a) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Write down the cofactor expansion of $\det(A)$ along the second column.

(b) Let $A = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3]$ and $B = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$ be two 3×3-matrices. Suppose that $\det(A) = 5$ and $\boldsymbol{b}_1 = \boldsymbol{a}_1$, $\boldsymbol{b}_2 = \boldsymbol{a}_1 + 2\boldsymbol{a}_2$, $\boldsymbol{b}_3 = \boldsymbol{a}_3$. What is $\det(B)$?

(c) Find $\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$.

(d) Find $\det \begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

Problem 6. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of A, as well as a basis for the corresponding eigenspaces.
(b) Diagonalize A. (That is, write $A = PDP^{-1}$ where D is diagonal.)

Problem 7. Consider the vector space

$$V = \{f : \mathbb{R} \to \mathbb{R} \ : \ f \text{ is 7-periodic and } f \text{ is “nice”}\}.$$

Here, f “nice” means, for instance, that f should be piecewise continuous or (more generally) that $\int_0^7 f(t)^2 dt$ should be finite.

(a) What is a natural inner product on V?
(b) Consider the 7-periodic function $f(t)$ with $f(t) = \begin{cases} 1, & \text{for } 0 \le t < 3, \\ 2, & \text{for } 3 \le t < 7. \end{cases}$
Compute the orthogonal projection of $f(t)$ onto the span of $\cos\left(3\frac{2\pi t}{7}\right)$.

Problem 8. Consider the space $\mathbb{P}^2$ of polynomials of degree up to 2, together with the inner product

$$\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t)\mathrm{d}t.$$

(a) Is the standard basis $1, t, t^2$ an orthogonal basis?
(b) Apply Gram–Schmidt to $1, t, t^2$ to obtain an orthonormal basis of $\mathbb{P}^2$.
(c) What is the orthogonal projection of t^2 onto $\text{span}\{1, t\}$?

SHORT ANSWERS

Note: On the actual exam all short answer question will be multiple choice. You will be entering your answers to the multiple choice questions on a scantron sheet that will be included with your exam. So please bring a **Number 2 pencil** to the exam. Thanks.

Short Problem 1. If A and B are 3×3 matrices with $\det(A) = 4$ and $\det(B) = -1$. What is the determinant of $C = 2A^T A^{-1} B A$?

Short Problem 2. If A is an $n \times n$ matrix, and S is an invertible $n \times n$ matrix. Are the characteristic polynomial of A and SAS^{-1} equal? The determinant?

Short Problem 3. Let A be a 7×7 matrix with $\dim \operatorname{Nul}(A) = 1$. What can you say about $\det(A)$?

Short Problem 4. Consider $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

(a) Using that the columns of A are orthogonal, find A^{-1}.
(b) Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_4$ be the columns of A. Without solving equations, find coefficients $c_1, \ldots, c_4$ such that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = c_1 \boldsymbol{w}_1 + \ldots + c_4 \boldsymbol{w}_4$.

Short Problem 5. Let A be a $n \times n$ matrix with $A^T = A^{-1}$. What can you say about $\det(A)$?

Short Problem 6. Let A be an $n \times n$ matrix with eigenvalue λ. Determine whether each of the following statements is correct.

(a) λ^2 is an eigenvalue of A^2.
(b) λ^{-1} is an eigenvalue of A^{-1}.
(c) $\lambda + 1$ is an eigenvalue of $A + I$.
(d) λ cannot be zero.

Short Problem 7. Consider a matrix

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \star \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \star \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \star \end{bmatrix},$$

in which the third column has not been specified, yet. Which of the following vectors can be added as a third column of Q such that Q is orthogonal?

(a) $\begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$,

(b) $\begin{bmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix}$,

(c) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

(d) none of the above.

Short Problem 8. True or false?

(a) If $A^T A$ is diagonal, then A has orthogonal columns.
(b) If A is an orthogonal matrix, then A^T is an orthogonal matrix.
(c) If $A\boldsymbol{x} = 0$, then $\boldsymbol{x}$ is orthogonal to the columns of A.
(d) For all $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.
(e) For all $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$.
(f) Every orthonormal set of vectors is linearly independent.
(g) Every subspace of $\mathbb{R}^n$ has an orthogonal basis.
(h) If every row of A adds up to 0, then $\det(A) = 0$.
(i) If every row of A adds up to 1, then $\det(A) = 1$.
(j) If A is invertible and B is not invertible, then AB is invertible.
(k) The determinant of A is the product of the diagonal entries of A.

Short Problem 9. Suppose that the projection matrix corresponding to orthogonal projection onto V is $P = \frac{1}{30}\begin{bmatrix} 29 & -2 & 5 \\ -2 & 26 & 10 \\ 5 & 10 & 5 \end{bmatrix}$.

(a) Is $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ in V?

(b) Find the vector in V which is closest to $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(c) What is the dimension of V?