Math 415 - Midterm 2

Thursday, October 23, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. **[1 point]** Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Babak, Mahmood, Michael, Nathan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	Shorts	$\sum$
/1	/10	/14	/10	/10	/25	/70

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of 8 pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + 2b - c = 0 \right\}$.

(a) [**5 points**] Find a basis for V.
(b) [**5 points**] Find a basis for the orthogonal complement of V.

Solution. (a) Observe that $V = \operatorname{Nul}\left(\begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}\right)$. The matrix is already in reduced echelon form, so we read off that a basis for V is given by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) The orthogonal complement of V is $V = \operatorname{Col}\left(\begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}^T\right)$. Hence, a basis is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 2 & 6 & 9 & 7 & 0 \\ -2 & -6 & 6 & 8 & 2 \end{bmatrix}.$$

(a) **[7 points]** Find a basis for $\operatorname{Nul}(A)$.
(b) **[5 points]** Find a basis for $\operatorname{Col}(A)$.
(c) **[2 points]** Determine the dimension of $\operatorname{Col}(A^T)$ and the dimension of $\operatorname{Nul}(A^T)$.

Solution. We start with row reducing A:

$$\begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 2 & 6 & 9 & 7 & 0 \\ -2 & -6 & 6 & 8 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 12 & 12 & 2 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) With x_2 and x_4 as free variables, we find the basis

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

(b) By selecting pivot columns of the original matrix, we find the basis

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

[Since these are 3 vectors, it follows that $\operatorname{Col}(A) = \mathbb{R}^3$. Note that, in general, it is not possible to select pivot columns of the echelon form to get a basis of $\operatorname{Col}(A)$. In this case, it is coincidence that the pivot columns of the echelon form B do give a basis for $\operatorname{Col}(A)$ (the reason is that $\dim \operatorname{Col}(A) = \dim \operatorname{Col}(B)$, and so $\operatorname{Col}(B) = \mathbb{R}^3$ as well). But giving the basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\operatorname{Col}(A)$ needs justification in order to get credit!]

(c) $\dim \operatorname{Col}(A^T) = \dim \operatorname{Col}(A) = 3$.
$\dim \operatorname{Nul}(A^T) = 3 - \dim \operatorname{Col}(A^T) = 0$.

Problem 3. **[10 points]** Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}.$$

Find the matrix A which represents T with respect to the following bases:

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \text{ of } \mathbb{R}^2, \text{ and } \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-2\end{bmatrix} \text{ of } \mathbb{R}^4.$$

Solution.

$$\begin{aligned}
T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) &= \begin{bmatrix}1\\-1\\0\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\\-2\end{bmatrix}\\
T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\\2\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}1\\1\\0\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\-1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\\-2\end{bmatrix}
\end{aligned}$$

Hence, the matrix A which represents T with respect to the given bases is

$$\begin{bmatrix}1 & 1\\1 & -1\\0 & 0\\-1 & -1\end{bmatrix}.$$

Problem 4. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

(a) [**4 points**] Draw a directed graph with numbered edges and nodes, whose edge-node incidence matrix is A.

(b) [**3 points**] Use a property of the graph to find a basis for $\mathrm{Nul}(A)$.

(c) [**3 points**] Use a property of the graph to find a basis for $\mathrm{Nul}(A^T)$.

Solution. (a) Left to your imagination!

(b) The graph is connected, so $\mathrm{Nul}(A)$ is 1-dimensional with basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) The graph has two independent loops: $\mathrm{edge}_1, \mathrm{edge}_2, \mathrm{edge}_3$ and $\mathrm{edge}_4, \mathrm{edge}_5, \mathrm{edge}_6$. Hence, a basis for $\mathrm{Nul}(A^T)$ is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

SHORT ANSWERS

[25 points overall, 2-4 points each]

Instructions: The following problems have a short answer. No reason needs to be given. If the problem is multiple choice, circle the correct answer (there is always exactly one correct answer).

Short Problem 1. **[2 points]** Let H be a subspace of $\mathbb{R}^7$ with basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$. What is the dimension of H?

Solution. 4

Short Problem 2. **[2 points]** Find a vector that is orthogonal to both $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Short Problem 3. **[2 points]** Let A be a matrix, and let B be its row reduced echelon form. Which of the following is true for any such matrices?

(a) $\text{Nul}(A) = \text{Nul}(B)$ and $\text{Nul}(A^T) = \text{Nul}(B^T)$
(b) $\text{Nul}(A) = \text{Nul}(B)$ and $\text{Nul}(A^T) \neq \text{Nul}(B^T)$
(c) $\text{Nul}(A) \neq \text{Nul}(B)$ and $\text{Nul}(A^T) = \text{Nul}(B^T)$
(d) $\text{Nul}(A) \neq \text{Nul}(B)$ and $\text{Nul}(A^T) \neq \text{Nul}(B^T)$
(e) None of these are true for all such matrices.

Solution. We always have $\text{Nul}(A) = \text{Nul}(B)$ because the null space is preserved by row operations.

On the other hand, $\text{Nul}(A^T)$ and $\text{Nul}(B^T)$ are usually different. However, they can, by chance, be equal (take a zero matrix, for instance).

Hence, the correct answer is (e) (but (b) received full credit as well).

Short Problem 4. **[2 points]** Let A be a 4×3 matrix with $\dim \text{Col}(A) = 3$. Which of the following is correct?

(a) The equation $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions.
(b) The equation $A\boldsymbol{x} = \mathbf{0}$ has exactly one solution.
(c) The columns of A are linearly dependent.
(d) The rows of A are linearly independent.
(e) None of the above is correct for all such A.

Solution. There are 3 columns and $\dim \text{Col}(A) = 3$, so the columns are linearly independent. Hence, $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, and the correct answer is (b).

Short Problem 5. [**2 points**] Let V be the following subspace of $\mathbb{R}^5$.

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : 3x_1 - x_2 = 0, \quad 3x_1 - x_3 = 0 \right\}$$

What is the dimension of V?

Solution. The two equations are clearly independent. Hence, $\dim(V) = 5 - 2 = 3$.

Short Problem 6. [**3 points**] Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2, and let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(p(t)) = p(t) + 2p'(t).$$

Which matrix A represents T with respect to the standard bases?
(Recall that the standard basis for $\mathbb{P}_2$ is given by $1, t, t^2$.)

Solution.

$$\begin{aligned} T(1) &= 1 \\ T(t) &= 2 + t \\ T(t^2) &= t^2 + 4t \end{aligned}$$

Hence, the matrix A representing T with respect to the standard bases is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Short Problem 7. Let a, b be in $\mathbb{R}$. Consider the three vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ b \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}.$$

[**2 points**] For which values of a and b are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ independent?

(a) $a = 0$ and $b = 4$
(b) $a \neq 0$ and $b \neq 4$
(c) $a = 0$ and $b \neq 4$
(d) $a \neq 0$ and $b = 4$

[**2 points**] For which values of a and b does $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ have dimension 1?

(a) $a = 0$ and $b = 4$
(b) $a \neq 0$ and $b \neq 4$
(c) $a = 0$ and $b \neq 4$
(d) $a \neq 0$ and $b = 4$

Solution. (b), and then (a).

Short Problem 8. [**2 points**] What is the dimension of the orthogonal complement of

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}?$$

Solution. The span is 1-dimensional in $\mathbb{R}^3$, hence the orthogonal complement has dimension 2.

Short Problem 9. [**2 points**] Consider the two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Which of the following is correct?

(a) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(b) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$
(c) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(d) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$

Solution. (c)

Short Problem 10. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

(a) [**1 point**] What is the dimension of $\text{Nul}(A)$?
(b) [**1 point**] What is the dimension of $\text{Col}(A)$?
(c) [**1 point**] What is the dimension of $\text{Nul}(A^T)$?
(d) [**1 point**] What is the dimension of $\text{Col}(A^T)$?

Solution. 2, 4, 0, 4