Math 415 - Midterm 2

Thursday, October 23, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
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Name:

NetID:

UIN:

Problem 0. ***[1 point]*** Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Babak, Mahmood, Michael, Nathan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

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/1	/?	/?	/?	/?	/?	/?	/?	/?

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

(a) Find a basis for Nul(A).
(b) Find a basis for Col(A^T).
(c) Determine the dimension of Col(A) and the dimension of Nul(A^T).

Solution:

(a) We have to transform A into the row reduced echelon form:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - R2, R1 \to R1 + 2R2, R2 \to -R2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have two free variables, x_3 and x_4. From the row reduced echelon form we get:

$$x_1 = -x_4, \; x_2 = -x_3$$

Thus,

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 = -x_4, x_2 = -x_3 \right\} = \left\{ \begin{bmatrix} -x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) (Transpose of) nonzero rows of the reduced echelon form will form a basis for Col(A^T), i.e., a basis for Col(A^T) is:

$$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$$

Note: nonzero rows of any echelon form of A (not necessarily the row reduced echelon form) form a basis for Col(A^T).

(c) The dimension of Col(A) is equal to the number of pivot columns in an echelon form (in particular the row reduced echelon form) of A, So $dim(\text{Col}(A)) = 2$.
The dimension of Nul(A^T) is equal to the number of zero rows in an echelon form (in particular the row reduced echelon form) of A, So $dim(\text{Nul}(A^T)) = 1$.

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with

$$T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}2\\2\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Find the matrix A which represents T with respect to the following bases:

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \text{ for } \mathbb{R}^2, \quad \text{and} \quad \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix} \text{ for } \mathbb{R}^3.$$

Solution: We have:

$$T(\begin{bmatrix}1\\1\end{bmatrix}) = T(\frac{1}{2}\begin{bmatrix}2\\0\end{bmatrix} - \begin{bmatrix}0\\-1\end{bmatrix}) = \frac{1}{2}\begin{bmatrix}2\\2\\2\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$T(\begin{bmatrix}1\\-1\end{bmatrix}) = T(\frac{1}{2}\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}) = \frac{1}{2}\begin{bmatrix}2\\2\\2\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & 2\\ -1 & -1\end{bmatrix}$$

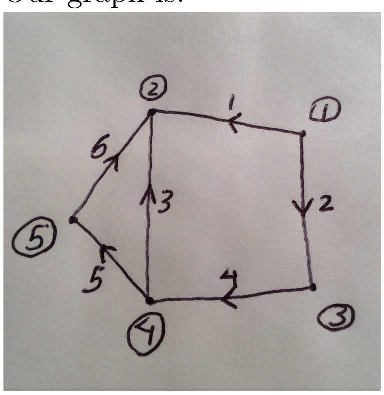
Problem 3. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

(a) Draw a directed graph with numbered edges and nodes, whose edge-node incidence matrix is A.
(b) Find a basis for $\mathrm{Col}(A^T)$ by choosing a spanning tree of this graph. (This question is not relevant for the second midterm exam!)
(c) Use a property of the graph (briefly explain!) to find a basis for $\mathrm{Nul}(A)$.
(d) Use a property of the graph (briefly explain!) to find a basis for $\mathrm{Nul}(A^T)$.

Solution:

(a) Our graph is:



(Note that the position of vertices does not matter at all; it only matters in which way they are connected.)

(b) Rows r_1, r_2, r_4, and r_5 correspond to a spanning tree of the graph. So a basis for $\mathrm{Col}(A^T)$ is:

$$\{r_1^T, r_2^T, r_4^T, r_5^T\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) Since the graph is connected, a basis for $\mathrm{Nul}(A)$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) The graph has two independent loops: $\text{edge}_1, -\text{edge}_3, -\text{edge}_4, -\text{edge}_2$ as well as $\text{edge}_3, -\text{edge}_6, -\text{edge}_5$. A basis for $\text{Nul}(A^T)$ is given by the corresponding vectors:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Problem 4. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - b = c \right\}$.

(a) Write V as a span.

(b) Find a basis for the orthogonal complement of V.

Solution:

(a) We have:

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - b = c \right\} = \left\{ \begin{bmatrix} b+c \\ b \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then we have $V = Col(A^T)$. The orthogonal complement of V ($= Col(A^T)$) is equal to Nul(A). We have

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \to R1 + R2, R2 \to -R2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the row reduced echelon form of A, we have:

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 = -x_3, x_2 = x_3, x_4 = 0 \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Therefore, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the orthogonal complement of Col(A^T).

(Alternative quick solution, after a moment of thought: V is defined to contain all vectors $[a, b, c, d]^T$ which are orthogonal to $[1, -1, -1, 0]^T$ (write out the inner product!). It follows that the orthogonal complement of V is spanned by precisely this vector, and so has basis given by $[1, -1, -1, 0]^T$.)

Problem 5. Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2, and let V be the subspace of polynomials $p(t)$ with the property that

$$\int_0^2 p(t)\mathrm{d}t = 0.$$

Find a basis for V. [*Hint*: Write $p(t) = a + bt + ct^2$ and use the integral condition to get a condition on the coefficients of $p(t)$.]

Solution: Note that, for $p(t) = a + bt + ct^2$, the integral condition becomes

$$\int_0^2 p(t)\mathrm{d}t = \int_0^2 (a + bt + ct^2)\mathrm{d}t = 2a + 2b + \frac{8}{3}c = 0.$$

Equivalently, $c = -\frac{3}{4}a - \frac{3}{4}b$. Hence, the polynomials in V are of the form

$$p(t) = a + bt - \tfrac{3}{4}(a + b)t^2 = a(1 - \tfrac{3}{4}t^2) + b(t - \tfrac{3}{4}t^2).$$

In other words, $V = \mathrm{span}\{1 - \frac{3}{4}t^2, t - \frac{3}{4}t^2\}$. Since the two polynomials $1 - \frac{3}{4}t^2, t - \frac{3}{4}t^2$ are not multiples of each other, they are independent and so form a basis for V.

Problem 6. Let $\mathbb{P}_3$ be the vector space of all polynomials of degree up to 3, and let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by

$$T(p(t)) = tp'(t) - 2p(t).$$

(a) Which matrix A represents T with respect to the standard bases?
(b) Find a basis for the null space of A.
[*Optional but recommended*: Interpret in terms of polynomials!]

Solution:

(a) We have:

$$\begin{aligned} T(1) &= t.0 - 2.1 = -2.1 + 0.t + 0.t^2 + 0.t^3 \\ T(t) &= t.1 - 2.t = 0.1 - 1.t + 0.t^2 + 0.t^3 \\ T(t^2) &= t.2t - 2.t^2 = 0.1 + 0.t + 0.t^2 + 0.t^3 \\ T(t^3) &= t.3t^2 - 2.t^3 = 0.1 + 0.t + 0.t^2 + 1.t^3 \end{aligned}$$

Therefore, the matrix A that represents T with respect to the standard bases is: (we put coefficients of $T(1), T(t), T(t^2)$, and $T(t^3)$ respectively in the first, second, third, and forth column)

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) A is already in row reduced echelon form. Hence, we have:

$$\text{Nul}(A) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 0, b = 0, d = 0 \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus, the set of all polynomials, p, so that $T(p) = 0$ is:

$$\left\{ a + bt + ct^2 + dt^3 \ : \ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ is in the Nul}(A) \right\} = \{ct^2 \ : \ c \in \mathbb{R}\}$$

Note that this means that the polynomials ct^2 are the only polynomials of degree up to 3 which solve the (differential) equation $tp'(t) - 2p(t) = 0$.

Problem 7. Let a, b be in $\mathbb{R}$. Consider the three vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} b \\ 1 \\ b \end{bmatrix}.$$

(a) For which values of a and b is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ a basis of $\mathbb{R}^3$?
(b) For which values of a and b does $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ have dimension 2?

Solution:

(a) $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis if they are linearly independent (since the dimension of $\mathbb{R}^3$ is equal to 3). To find out, we transform $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ into echelon form. We find:

$$A = \begin{bmatrix} 1 & 1 & b \\ 1 & a & 1 \\ b & 0 & b \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 - bR1} \begin{bmatrix} 1 & 1 & b \\ 0 & a-1 & 1-b \\ 0 & -b & b-b^2 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{b}{a-1} R2} \begin{bmatrix} 1 & 1 & b \\ 0 & a-1 & 1-b \\ 0 & 0 & (b-b^2)\frac{a}{a-1} \end{bmatrix}$$

Note, however, that the last step is only OK if $a \neq 1$. In that case, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent if all columns are pivot columns, i.e., $b(1-b)\frac{a}{a-1} \neq 0$. Therefore, if $a \neq 1$, then the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis for all values of a, b that satisfy $a \neq 0$, $b \neq 0$ and $b \neq 1$.

We still need to consider the case $a = 1$:

$$A = \begin{bmatrix} 1 & 1 & b \\ 1 & 1 & 1 \\ b & 0 & b \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 - bR1} \begin{bmatrix} 1 & 1 & b \\ 0 & 0 & 1-b \\ 0 & -b & b-b^2 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & b \\ 0 & -b & b-b^2 \\ 0 & 0 & 1-b \end{bmatrix}$$

Hence, if $a = 1$, the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent if all columns are pivot columns, i.e., $1 - b \neq 0$ and $b \neq 0$.

In conclusion, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis as long as $a \neq 0$, $b \neq 0$ and $b \neq 1$.

(b) Since we have at least two pivot positions in an echelon form (why?), $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ has dimension at least two; and the dimension is equal to two if we have two pivot positions, i.e., $b = 0$, or $b = 1$. Therefore, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ has dimension 2 for all values of a, b so that $a = 0$, or $b = 0$, or $b = 1$.

Note: an easier way to see all this is to use one column operation (equivalent to using row operation on A^T). We have:

$$A = \begin{bmatrix} 1 & 1 & b \\ 1 & a & 1 \\ b & 0 & b \end{bmatrix} \xrightarrow{C1 \to C1 - C3} \begin{bmatrix} 1-b & 1 & b \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$$

(Note that such an elementary column operation does not change the column space.) There are 3 pivot positions if and only if $a, b \neq 0$ and $b \neq 1$.

Problem 8. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ -1 & -1 & -3 \end{bmatrix}$.

(a) Under which condition(s) on $\boldsymbol{b}$ has the system $A\boldsymbol{x} = \boldsymbol{b}$ a solution?
(b) Find a basis for Nul(A).
(c) Note that $A \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix}$. Find all solutions to $A\boldsymbol{x} = \begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix}$.

Solution:

(a) $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is in Col(A). We transform A into echelon form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ -1 & -1 & -3 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1, R3 \to R3 + R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The first column and the second column are pivot columns, so we have:

$$\text{Col}(A) = \mathit{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right\}$$

[Note (added later on): That $\boldsymbol{b}$ should be in Col(A) is a correct "condition" but not a very explicit one (it still requires work to check whether it satisfied). Here is the condition I was expecting: in order for the system to have a solution, $\boldsymbol{b}$ has to be orthogonal to Nul(A^T). The RREF of A^T is:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 0 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We thus find that the left nullspace is spanned by $[3, -1, 1]^T$. Since $\boldsymbol{b}$ has to be orthogonal to this vector, we find that the system has a solution if and only if $3b_1 - b_2 + b_3 = 0$.]

(b) We transform A into the row reduced echelon form: (we can continue from the echelon form that we got in first part)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, we have:

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = -5x_3, x_2 = 2x_3 \right\} = \left\{ \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(c) Recall that $\{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}\} = \text{Nul}(A) + \boldsymbol{b}_0$ where $\boldsymbol{b}_0$ is a specific solution of $A\boldsymbol{x} = \boldsymbol{b}$. Therefore, we have:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix} \right\} = \text{Nul}(A) + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\} + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

SHORT ANSWERS

[?? points overall, 3 points each]

Instructions: The following problems have a short answer. No reason needs to be given. If the problem is multiple choice, circle the correct answer (there is always exactly one correct answer).

Short Problem 1. Give a precise definition of what it means for vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ to be linearly independent.

Solution: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent if and only if $x_1\boldsymbol{v}_1 + \ldots + x_n\boldsymbol{v}_n = 0$ has exactly one solution (i.e. the trivial solution, all x_i's equal to zero).

Short Problem 2. Write down a basis for the orthogonal complement of $W = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

Solution: $W = \text{Col}(\begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix})$ so the orthogonal complement of W is $\text{Nul}(\begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}^T) = \text{Nul}(\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\end{bmatrix})$.

$$\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

Thus,

$$\text{Nul}(A) = \text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

and $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for orthogonal complement of W.

Short Problem 3. Let A be an 4×3 matrix, whose row space has dimension 2. What is the dimension of $\text{Nul}(A)$?

Solution: We have three variables and two pivot positions, so the number of free variables is equal to 1. So $\dim(\text{Nul}(A)) = 1$, number of free variables.

Short Problem 4. Let A be an 3×3 matrix, whose column space has dimension 3. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$, what can you say about the number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$?

Solution: The dimension of the column space of A is 3, so we have 3 pivot columns in the reduced echelon form (i.e. each column is a pivot column). Thus, A is invertible and $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for each $\boldsymbol{b}$.

Short Problem 5. Let A be an 3×5 matrix of rank 2. Is it possible to find two linearly independent vectors that are orthogonal to the column space of A? For the row space?

(a) Possible for both.
(b) Possible only for the column space.
(c) Possible only for the row space.
(d) Not possible in either case.
(e) Not enough information to decide.

Solution: The dimension of the column space of A is 2 and it is a subset of $\mathbb{R}^3$ (each column has 3 entries), so the dimension of the orthogonal complement of $\text{Col}(A)$ is $1 = 3 - 2$. Thus, it is not possible to find two linearly independent vectors that are orthogonal to the column space of A.
The dimension of the row space of A is 2 and it is a subset of $\mathbb{R}^5$ (each row has 5 entries), so the dimension of the orthogonal complement of $\text{Col}(A^T)$ is $3 = 5 - 2$. Thus, it is possible to find two linearly independent vectors that are orthogonal to the row space of A.
Therefore, it is possible only for the row space.

Short Problem 6. Let A be a matrix, and let B be its row reduced echelon form. Which of the following is true for any such matrices?

(a) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(b) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$
(c) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(d) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$
(e) None of these are true for all such matrices.

Solution:

(a) False, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{Col}(A) \neq \text{Col}(B)$.
(b) False, we always have $\text{Col}(A^T) = \text{Col}(B^T)$.
(c) False, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{Col}(A) = \text{Col}(B)$.
(d) False, we always have $\text{Col}(A^T) = \text{Col}(B^T)$.

Note: We always have $\text{Col}(A^T) = \text{Col}(B^T)$, but we do not always have $\text{Col}(A) = \text{Col}(B)$. However, for special matrices, we might have $\text{Col}(A) \neq \text{Col}(B)$ by coincidence.

Short Problem 7. Let A be a 5×4 matrix. Suppose that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has the solution set

$$\left\{ \begin{bmatrix} 1 - c + d \\ c \\ 3 - 2d \\ d \end{bmatrix} : c, d \text{ in } \mathbb{R} \right\}.$$

(a) Give a basis for the null space of A.
(b) What is the rank of A?

Solution:

(a) We can write the set of solutions of $A\boldsymbol{x} = \boldsymbol{b}$ as:

$$\left\{ \begin{bmatrix} 1 - c + d \\ c \\ 3 - 2d \\ d \end{bmatrix} : c, d \text{ in } \mathbb{R} \right\} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Thus,

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Therefore, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A).

(b) The rank of A is equal to $2 = 4 - 2$ (the number of columns of A - the dimension of Nul(A)).

Short Problem 8. The linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if ...

(a) $\boldsymbol{b}$ is orthogonal to Col(A).
(b) $\boldsymbol{b}$ is orthogonal to Col(A^T).
(c) $\boldsymbol{b}$ is orthogonal to Nul(A).
(d) $\boldsymbol{b}$ is orthogonal to Nul(A^T).
(e) Neither of these guarantees a solution.

Solution: $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is in Col(A). On the other hand, the orthogonal complement of Col(A) is Nul(A^T). Therefore, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is orthogonal to Nul(A^T).

Short Problem 9. Let A be the edge-node incidence matrix of a directed graph. Suppose that this graph is not a tree (that is, the graph contains at least one loop). What can you say about the rows of A?

Solution: This implies that Nul(A^T) has at least dimension 1. Hence, the rows of A are not linearly independent (recall that elements in Nul(A^T) correspond to linear relations between the columns of A^T).