Math 415 - Midterm 1

Thursday, September 25, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. [**1 point**] Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Babak, Mahmood, Michael, Nathan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	5	Shorts	$\sum$
/1	/11	/15	/15	/12	/8	/21	/83

Good luck!

Problem 1. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) **[8 points]** Determine A^{-1}.

(b) **[3 points]** Using A^{-1}, solve $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution 1. (a) Using the Gauss–Jordan method, we find:

$$\left(\begin{array}{cccccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{R1 \leftrightarrow R3}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\overset{R2 \to R2 - 2R1}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\overset{R3 \to R3 + R2}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 & 1 & -2 \end{array}\right)$$

$$\overset{\substack{R2 \to -R2 \\ R3 \to -R3}}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

$$\overset{\substack{R1 \to R1 - R3 \\ R2 \to R2 - 2R3}}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

$$\overset{R1 \to R1 - R2}{\rightsquigarrow} \left(\begin{array}{cccccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

Hence,

$$A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}.$$

(b) We obtain

$$\boldsymbol{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

[Note that this is actually obvious when thinking in terms of the column picture of the linear system.]

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

(a) **[10 points]** Calculate the LU decomposition of A.

(b) **[5 points]** Solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$$

without reducing the augmented matrix, but using the LU decomposition.

Solution 2. (a) We first find U by Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} \overset{\substack{R2 \to R2 - R1 \\ R3 \to R3 - R1}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{pmatrix}$$

$$\overset{R3 \to R3 - R2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

We determine the matrix L from the row operations performed to get the LU decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

(b) We solve $A\boldsymbol{x} = \boldsymbol{b}$ by first solving $L\boldsymbol{c} = \boldsymbol{b}$ via forward substitution. We find

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{c} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{c} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$

Finally, we solve $U\boldsymbol{x} = \boldsymbol{c}$ via backward substitution.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Problem 3. Consider the following system of linear equations:

$$\begin{array}{rrrrcl} x_1 & -2x_2 & +x_3 & & = & 1 \\ -x_1 & +2x_2 & +x_3 & +2x_4 & = & 1 \\ -2x_1 & +4x_2 & +4x_3 & +6x_4 & = & 4 \end{array}$$

(a) **[2 points]** Write down the augmented matrix corresponding to this system.
(b) **[7 points]** Determine the row reduced echelon form of the augmented matrix.
(c) **[6 points]** Use your result in (b) to find a parametric description of the set of solutions to the system of linear equations.

Solution 3. (a) The augmented matrix is

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 2 & 1 & 2 & 1 \\ -2 & 4 & 4 & 6 & 4 \end{pmatrix}.$$

(b) Gauss–Jordan elimination produces:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 2 & 1 & 2 & 1 \\ -2 & 4 & 4 & 6 & 4 \end{pmatrix} \overset{\substack{R2\to R2+R1 \\ R3\to R3+2R1}}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 6 & 6 & 6 \end{pmatrix}$$

$$\overset{R3\to R3-3R2}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R2\to \frac{1}{2}R2}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R1\to R1-R2}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, the row reduced echelon form is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) The variables x_2 and x_4 are free, and we find the parametric description of the solutions as follows:

$$\begin{cases} x_1 = 2x_2 + x_4 \\ x_2 \text{ is free} \\ x_3 = 1 - x_4 \\ x_4 \text{ is free} \end{cases}$$

Optionally, and equivalently, we can write the set of solutions as

$$\left\{ \begin{pmatrix} 2x_2 + x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{pmatrix} : x_2, x_4 \text{ in } \mathbb{R} \right\} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

[Note that the final span is the null space of the coefficient matrix.]

Problem 4. Let

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ h \\ 3h \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) **[8 points]** For which value of h is $\boldsymbol{w}$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(b) **[4 points]** For the value of h found in (a), write down the linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ which gives $\boldsymbol{w}$.

Solution 4. (a) $\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ if and only if the system with augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & 3 & 3h \end{pmatrix}$$

is consistent. From the echelon form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & 3 & 3h \end{pmatrix} \overset{\substack{R2\to R2-R1 \\ R3\to R3-R1}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 2 & 3h-1 \end{pmatrix} \overset{R3\to R3-2R2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 0 & h+1 \end{pmatrix}$$

we find that the system is consistent if and only if $h = -1$.

Hence, $\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ if and only if $h = -1$.

(b) The coefficients of such a linear combinations are given by the solutions of the system. In the case $h = -1$, this system has augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 0 & h+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple backward substitution gives the solution $x_2 = -2$ and $x_1 = 3$. The corresponding linear combination is

$$3\boldsymbol{v}_1 - 2\boldsymbol{v}_2 = \boldsymbol{w}.$$

Problem 5. **[8 points]** Determine which of the following sets are a subspace of the vector space of all 2×2 matrices. In each case, give a short reason.

(a) $W_1 = \left\{ \begin{bmatrix} 2a & b \\ b & 3a \end{bmatrix} \ : \ a, b \text{ in } \mathbb{R} \right\}$

(b) $W_2 = \left\{ \begin{bmatrix} 2a & b \\ b & 3a \end{bmatrix} \ : \ a, b \text{ in } \mathbb{R} \text{ and } a + b = 1 \right\}$

Solution 5. (a) W_1 is a subspace because we can write it as a span:

$$W_1 = \text{span}\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

(b) W_2 is not a subspace because it does not contain the zero vector. Indeed, if

$$\begin{pmatrix} 2a & b \\ b & 3a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then $a = 0$ and $b = 0$. But this contradicts $a + b = 1$.

SHORT ANSWERS

[21 points overall, 3 points each]

Instructions: The following problems have a short answer. No reason needs to be given. If the problem is multiple choice, circle the correct answer (there is always exactly one correct answer).

Short Problem 1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. Compute $A^T A$.

$$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Short Problem 2. Let A be a matrix such that, for every $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ x \\ x - z \end{bmatrix}$.

Then, what is A?

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Short Problem 3. Let C be a 3×4 matrix such that C has two pivot columns, and let $\boldsymbol{d}$ be a vector in $\mathbb{R}^3$. Is it true that, if the equation $C\boldsymbol{x} = \boldsymbol{d}$ has a solution, then it has infinitely many solutions?

(a) True.
(b) False.
(c) Unable to determine.

This is true, because there is a free variable (actually, it is two). Hence, the system has infinitely many solutions unless it is inconsistent.

Short Problem 4. Let

$$A = \begin{bmatrix} a & a+1 \\ a+1 & a \end{bmatrix}.$$

For which choice(s) of a is the matrix A *not* invertible?

A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not invertible if and only if $ad - bc = 0$. Since $a^2 - (a+1)^2 = -2a - 1$, A is not invertible if and only if $a = -\frac{1}{2}$.

Short Problem 5. There is one vector which every subspace of $\mathbb{R}^2$ has to contain. Which vector is that?

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Short Problem 6. Let W_1 be the set of all polynomials $p(t)$ which have a zero at $t = 1$ (that is, $p(1) = 0$), and let W_0 be the set of all polynomials $p(t)$ which have a zero at $t = 0$. Are these sets subspaces of the vector space of all polynomials?

(a) Both W_0 and W_1 are subspaces.
(b) Only W_0 is a subspace.
(c) Only W_1 is a subspace.
(d) Neither W_0 nor W_1 are subspaces.

If two polynomials have a zero at $t = t_0$ for some fixed t_0 in $\mathbb{R}$, then the sum of these polynomials has a zero at $t = t_0$ as well. Likewise, for scaling. Hence, both W_0 and W_1 are subspaces.

Short Problem 7. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Which of the following is true?

(a) W is empty.
(b) W is a line.
(c) W is a plane.
(d) W is all of $\mathbb{R}^3$.

This is a plane.