Math 415 - Midterm 1

Thursday, September 25, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. [**1 point**] *Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Babak, Mahmood, Michael, Nathan, Tigran, Travis).*

Section:	*TA:*

To be completed by the grader:

0	1	2	3	4	5	6	Shorts	$\sum$
/1	/15	/10	/15	/10	/10	/12	/21	/94

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of 9 pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. *Let*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) [**8 points**] *Determine* A^{-1}.
(b) [**2 points**] *Check whether* $AA^{-1} = I_3$.

Solution: We have:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - R1, R3 \to R3 - R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 + 2R3, R2 \to R2 - 2R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R3 \to -R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$.

Problem 2. *Consider the matrix*

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 1 & 2 \\ 6 & 7 & 5 \end{bmatrix}.$$

(a) **[10 points]** *Calculate the LU decomposition of A.*

(b) **[5 points]** *Solve*

$$\begin{bmatrix} 3 & 2 & 0 \\ 3 & 1 & 2 \\ 6 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

without reducing the augmented matrix, but using the LU decomposition.

Solution:

(a) First, we transform A to echelon form (an upper triangular matrix) using upward row operations:

$$\begin{bmatrix} 3 & 2 & 0 \\ 3 & 1 & 2 \\ 6 & 7 & 5 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ 6 & 7 & 5 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow{R3 \to R3 + 3R2} \begin{bmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 11 \end{bmatrix} = U$$

To get L, we have to apply the inverse of the row operations in the reverse order to I:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 - 3R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 + 2R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 + R1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = L$$

[Note: for the exam, you may also immediately write down L. The above details give you another chance to see why that is.]

(b) Let $\boldsymbol{c} = U\boldsymbol{x}$. First we solve $L\boldsymbol{c} = \boldsymbol{b}$:

$$\begin{aligned} c_1 &= 4 \\ c_1 + c_2 = 7 \Rightarrow c_2 &= 3 \\ 2c_1 - 3c_2 + c_3 = 10 \Rightarrow c_3 &= 11 \end{aligned}$$

Finally, we solve $U\boldsymbol{x} = \boldsymbol{c}$ to find $\boldsymbol{x}$:

$$\begin{aligned} 11x_3 = 11 \Rightarrow x_3 &= 1 \\ -x_2 + 2x_3 = 3 \Rightarrow x_2 &= -1 \\ 3x_1 + 2x_2 = 4 \Rightarrow x_1 &= 2 \end{aligned}$$

Thus:

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

[Note: make sure to quickly check your answer by computing $A\boldsymbol{x}$.]

Problem 3. *Let*

$$B = \begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix}.$$

(a) **[8 points]** *Determine the reduced echelon form of B.*

(b) **[7 points]** *Use your result in* (a) *to find a parametic description of the set of solutions of the following system of linear equations:*

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Solution:

(a) We have:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 + 3R1} \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Note that B is the augmented matrix corresponding to this system of linear equations. According to the reduced echelon form of B, x_1 and x_2 are dependent variables (since the first and the second column are pivot columns) and x_3 is free. We have:

$$\begin{aligned} x_1 + 4x_3 = -5 &\Rightarrow x_1 = -4x_3 - 5 \\ x_2 - 3x_3 = 3 &\Rightarrow x_2 = 3x_3 + 3 \end{aligned}$$

So the set of solutions is:

$$\left\{ \begin{bmatrix} -4x_3 - 5 \\ 3x_3 + 3 \\ x_3 \end{bmatrix} \ : \ x_3 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \ : \ x_3 \in \mathbb{R} \right\}$$

Which is a line that does not go through the origin.

Problem 4. [10 points] *Consider the vectors*

$$\boldsymbol{w} = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Is $\boldsymbol{w}$ *in* $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$*? Show your calculations!*

Solutions: We have to determine if there are real numbers x, y, and z such that $x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 = \boldsymbol{w}$. This is equivalent to determining if the following system of equations is consistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - 1/2R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R4 \to R4 + R3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this echelon form, there is no row of the form $[0\ 0\ 0 \mid r]$, where r is non-zero (or in other words, there is no pivot position in the right hand side). Thus, the system is consistent. Therefore, it is possible to write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, i.e., $\boldsymbol{w}$ is in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Problem 5. **[10 points]** *Let*

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ h_1 \\ h_2 \end{bmatrix}.$$

For which values of h_1 and h_2 is $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

Solution: Similar to the previous problem, we have to determine when the following system of equations is consistent:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & h_1 \\ -1 & 3 & h_2 \end{array}\right] \xrightarrow{R3 \to R3+R1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & h_1 \\ 0 & 3 & h_2+1 \end{array}\right] \xrightarrow{R3 \to R3+3R2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & h_1 \\ 0 & 0 & 3h_1+h_2+1 \end{array}\right]$$

The system is consistent if and only if $3h_1 + h_2 + 1 = 0$, i.e., $h_2 = -3h_1 - 1$.

Hence, $\boldsymbol{v}_3$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ if and only if $h_2 = -3h_1 - 1$.

Problem 6. [**12 points**] *Determine which of the following sets are subspaces and* give reasons*:*

(a) $W_1 = \{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ ab = 0\}$,

(b) $W_2 = \{ \begin{bmatrix} a+1 \\ a \end{bmatrix} \ : \ a \textit{ in } \mathbb{R}\}$,

(c) $W_3 = \{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a^2 + b^2 \leq 1\}$.

Solution:

(a) W_1 is not a subspace, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in W_1 but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in W_1.

(b) W_2 is not a subspace, since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W_2.

(c) W_3 is not a subspace, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W_3 but $2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in W_3.

SHORT ANSWERS

[21 points overall, 3 points each]

Instructions: The following problems have a short answer. No reason needs to be given. If the problem is multiple choice, circle the correct answer (there is always exactly one correct answer).

Short Problem 1. *Let A be a matrix such that, for every* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in* $\mathbb{R}^3$, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ 0 \\ 2x-z \end{bmatrix}$.

Then, what is A?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Short Problem 2. *Let* $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix}$. *Then, what is A^T?*

$$A^T = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Short Problem 3. *The set of solutions in $\mathbb{R}^3$ of the equation*

$$x_1 - 3x_2 + 2x_3 = 1$$

is

(a) *empty,*
(b) *a line not through the origin,*
(c) *a line through the origin,*
(d) *a plane.*

Solution: We have two free variables, x_2 and x_3, so it is a **plane**.

Short Problem 4. *Let A be an $l \times m$ matrix and B be an $n \times p$ matrix. Under which condition is $A^T B$ defined?*

A^T is an $m \times l$ matrix, so we should have $l = n$. [Note: the number of columns of the first matrix (A^T) should be the same as the number of rows of the second matrix (B)]

Short Problem 5. *Let C be a 3×4 matrix such that C has two pivot columns. Is it true that the equation $C\boldsymbol{x} = \boldsymbol{d}$ has a solution for every $\boldsymbol{d}$ in $\mathbb{R}^3$.*

(a) *True.*
(b) *False.*
(c) *Unable to determine.*

Solution: False. Since if we transform $[C|\boldsymbol{d}]$ into echelon form we get a row of zeros in the left hand side (the number of pivots is less than the number of rows) and we could have a non-zero element on the right hand side (rendering the system inconsistent). So in some cases there is no solution.

Note 1: We have more variables than the number of pivots so there is a free variable which means we have no or infinitely many solutions for each $\boldsymbol{d}$ in $\mathbb{R}^3$.

Note 2: If C has 3 pivot columns then the equation $C\boldsymbol{x} = \boldsymbol{d}$ has infinitely many solutions for each $\boldsymbol{d}$ in $\mathbb{R}^3$.

Short Problem 6. *Let*

$$A = \begin{bmatrix} 3 & a-6 \\ 3a & -a+6 \end{bmatrix}.$$

For which choices of a is the matrix A invertible?

Using the formula for the inverse of a 2×2 matrix, we get that A is invertible if and only if $3(-a+6) - 3(a-6)a = -3(a-6)(1+a) \neq 0$. Therefore, A is invertible if and only if $a \neq 6, -1$.

Short Problem 7. *How many solutions has a linear system with 4 equations and 5 unknowns?*

(a) *The system either has no solution or infinitely many solutions.*
(b) *The system has no solution.*
(c) *The system has exactly one solution.*
(d) *The system has infinitely many solution.*

Solution: The number of variables is more than the number of equations, so we will get at least one free variable. Thus, the system either has no solution (if inconsistent) or infinitely many solutions (if consistent).