Preparation problems for the discussion sections on November 11th and 13th

1. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

a. *Find the QR decomposition of A: write* $A = QR$ *where Q is a matrix with orthonormal columns and R is an upper triangular matrix.*

b. *Let* $\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *Use the QR decomposition of A to find the least squares solution of* $A\widehat{\boldsymbol{x}} = \boldsymbol{b}$ *(by solving* $R\widehat{\boldsymbol{x}} = Q^T\boldsymbol{b}$*).*

2. **a.** *Compare* $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and the "row flipped" determinant* $\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

b. *If* $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, *what is* $\det(A)$*?*

c. *If* $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix}$, *what is* $\det(A)$*?*

d. *If* $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$, *what is* $\det(A)$*?*

e. *If* A, B *are* 3×3 *matrices with* $\det(A) = 2$, $\det(B) = -1$, *calculate*

(i) $\det(BA^T)$,

(ii) $\det(BAB^{-1})$,

(iii) $\det(A^{-1})$.

f. *If* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, *find* $\det(A)$ *by expanding along the last column.*

3. **a.** *Someone tells you that* $\det$ *is linear, so* $\det(3A) = 3\det(A)$*. What do you answer? (What about* $\det(3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$*? If A is a* 3×3 *matrix, and* $\det(A) = 2$ *what is* $\det(3A)$*?)*

b. *Somebody tells you that the matrix*

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix}$$

is invertible. What do you say?

c. *Let*

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}.$$

Calculate $\det(A)$*. Is A invertible?*

d. *Let A be a 3×3 matrix so that* $A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$. *What is det$(A)$.*

4. *Reading through your favorite linear algebra textbook, you find the following interesting statement: if the columns of A are independent, then the orthogonal projection onto ColA has projection matrix $A(A^TA)^{-1}A^T$.*

a. *How does this formula simplify in the case when A has orthonormal columns?*

b. *Let* $Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col(Q)?*

c. *Let* $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col(Q)? Explain why your answer is not surprising.*

d. *(optional) Can you explain the formula $A(A^TA)^{-1}A^T$ for the projection matrix using the normal equations for least squares?*

5. *True or False? Justify your answers!*

a. *Let Q be a 3×3 orthogonal matrix. Then det$(Q) = 1$.*

b. *If det(A) =det$(B) = 0$ then det$(A + B) = 0$.*

c. *We say A and B ($n \times n$ matrices) are similar if $A = DBD^{-1}$ for an invertible matrix D. Let A and B be similar matrices, then det(A) =det(B).*

d. *Let A and B be 3×3 matrices. If det(A) =det(B) then A and B are similar. [Note: number of pivots in DBD^{-1} is equal to the number of pivots in B. (Why?) Use this fact to find a counter example.]*

e. *Let A be a 3×3 matrix so that det$(A) = 0$. Then $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for each vector $\boldsymbol{b}$.*

f. *Let A be a 3×3 matrix so that det$(A) = 9$. Then det$(2A) = 18$.*

g. *Let R be a 2×3 matrix. Then det$(R^TR) = 0$.*

h. *Let R be a 2×3 matrix. Then det$(RR^T) = 0$.*

6. *Let f be a function with period 2π that satisfies $f(x) = x$ on $(-\pi, \pi]$. Find the Fourier series of f.*