Preparation problems for the discussion sections on November 11th and 13th

1. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

a. *Find the QR decomposition of A: write $A = QR$ where Q is a matrix with orthonormal columns and R is an upper triangular matrix.*

b. *Let* $\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *Use the QR decomposition of A to find the least squares solution of $A\widehat{\boldsymbol{x}} = \boldsymbol{b}$ (by solving $R\widehat{\boldsymbol{x}} = Q^T\boldsymbol{b}$).*

Solution:

a. We start with columns of $A(= [\mathbf{v}_1\mathbf{v}_2])$ and we use Gram-Schmidt to find columns of $Q(= [\mathbf{q}_1\mathbf{q}_2])$:

$$q_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

and,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2 - (\mathbf{q}_1 \cdot \mathbf{v}_2)\mathbf{q}_1}{\|\mathbf{v}_2 - (\mathbf{q}_1 \cdot \mathbf{v}_2)\mathbf{q}_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \|} = \frac{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}}{\| \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \|} = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

We have:

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

b. We have to solve $R\widehat{\mathbf{x}} = Q^T\mathbf{b}$:

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Therefore, $\widehat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

2. **a.** *Compare* $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and the "row flipped" determinant* $\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

b. *If* $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, *what is* $\det(A)$*?*

c. *If* $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix}$, *what is* $\det(A)$*?*

d. *If* $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$, *what is* $\det(A)$*?*

e. *If* A, B *are* 3×3 *matrices with* $\det(A) = 2$, $\det(B) = -1$, *calculate*

(i) $\det(BA^T)$,

(ii) $\det(BAB^{-1})$,

(iii) $\det(A^{-1})$.

f. *If* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, *find* $\det(A)$ *by expanding along the last column.*

Solution:

a. We have:

$$\det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 1.4 - 2.3 = -2$$

and,

$$\det(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}) = 3.2 - 4.1 = 2$$

So, $\det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = -\det(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix})$. This agrees with the fact that we know that the interchange of two rows changes the sign of the determinant.

b. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R5, R2 \leftrightarrow R4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we swap rows twice, we have:

$$\det(A) = -(-\det(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix})) = 1$$

c. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1, R3 \to R3 - 3R1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & -6 \end{bmatrix}$$

Since the row operations that we used do not change the value of the determinant, we have:

$$\det(A) = \det(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & -6 \end{bmatrix}) = 1.0.(-6) = 0$$

d. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1, R3 \to R3 - 3R1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the row operations that we used do not change the value of the determinant, we have:

$$\det(A) = \det(\begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}) = 0$$

e. (i) $\det(BA^T) = \det(B)\det(A^T) = \det(B)\det(A) = -2$,
(ii) $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A)\frac{1}{\det(B)} = \det(A) = 2$,
(iii) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{2}$.

f. We have:

$$\det(A) = 3\det(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}) - 1\det(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}) + 3\det(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}) = 3.1 - 1.(-1) + 3.(-4) = -8$$

3. **a.** *Someone tells you that* det *is linear, so* $\det(3A) = 3\det(A)$. *What do you answer? (What about* $\det(3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$*? If A is a 3×3 matrix, and* $\det(A) = 2$ *what is* $\det(3A)$*?)*

b. *Somebody tells you that the matrix*

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix}$$

is invertible. What do you say?

c. *Let*

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}.$$

Calculate $\det(A)$. *Is A invertible?*

d. *Let A be a 3×3 matrix so that* $A\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$. *What is det($A$).*

Solution:

a. In general, if A is a $n \times n$ matrix then $\det(3A) = 3^n \det(A)$. In particular, if A is a 3×3 matrix, and $\det(A) = 2$ then $\det(3A) = 3^3 \det(A) = 27.2 = 54$.
Hence, if $\det(3A) = 3\det(A)$, then either $n = 1$ (i.e., A is a 1×1 matrix) or $\det A = 0$. Otherwise, the claim that $\det(3A) = 3\det(A)$ is false.

b. Since A has a column of zeros, $\det(A) = 0$. In other words, A is not invertible. We should tell the person to review their linear algebra.

c. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \xrightarrow{R2 \to R2-2R1\ ,\quad R3\to R3+R1} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$\xrightarrow{R4\to R4+2R3, R4\to R4+5/2R1} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Since the row operations that we used do not change the value of the determinant, we have:

$$\det(A) = \det(\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}) = 20$$

A is invertible since $\det(A) \neq 0$.

d. Since $A\mathbf{x} = 0$ has a non-zero solution, A is not invertible, i.e., $\det(A) = 0$.

4. *Reading through your favorite linear algebra textbook, you find the following interesting statement: if the columns of A are independent, then the orthogonal projection onto ColA has projection matrix $A(A^TA)^{-1}A^T$.*

a. *How does this formula simplify in the case when A has orthonormal columns?*

b. *Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix}$. What is the projection matrix corresponding to the orthogonal projection onto Col(Q)?*

c. *Let $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$. What is the projection matrix corresponding to the orthogonal projection onto Col(Q)? Explain why your answer is not surprising.*

d. *(optional) Can you explain the formula $A(A^TA)^{-1}A^T$ for the projection matrix using the normal equations for least squares?*

Solution:

a. If A has orthonormal columns then $A^TA = I$. So the projection matrix is:

$$A(A^TA)^{-1}A^T = AA^T$$

b. Since Q has orthonormal columns, the projection matrix is:

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{9}{25} & -\frac{12}{25} \\ 0 & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

c. Q has orthonormal columns and is square, so is orthogonal and satisfies $Q^{-1} = Q^T$. Therefore, the projection matrix QQ^T is equal to I.
Explanation: since the columns of Q are linearly independent and Q has 3 columns, the columns of Q form a basis for $\mathbb{R}^3$. In other words, $\text{Col}(Q) = \mathbb{R}^3$ and projection of each vector in $\mathbb{R}^3$ onto $\text{Col}(Q)$ is itself, i.e., the projection matrix is I.

5. *True or False? Justify your answers!*

a. *Let Q be a 3×3 orthogonal matrix. Then $\det(Q) = 1$.*

b. *If det(A) =det(B) $= 0$ then det($A+B$) $= 0$.*
c. *We say A and B ($n \times n$ matrices) are similar if $A = DBD^{-1}$ for an invertible matrix D. Let A and B be similar matrices, then det(A) =det(B).*
d. *Let A and B be 3×3 matrices. If det(A) =det(B) then A and B are similar. [Note: number of pivots in DBD^{-1} is equal to the number of pivots in B. (Why?) Use this fact to find a counter example.]*
e. *Let A be a 3×3 matrix so that det(A) $= 0$. Then $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for each vector $\boldsymbol{b}$.*
f. *Let A be a 3×3 matrix so that det(A) $= 9$. Then det($2A$) $= 18$.*
g. *Let R be a 2×3 matrix. Then det($R^T R$) $= 0$.*
h. *Let R be a 2×3 matrix. Then det(RR^T) $= 0$.*

Solution:

a. False, we have $QQ^T = I$ so $\det(Q)\det(Q^T) = \det(Q)^2 = \det(I) = 1$. Hence, $\det(Q) = 1$ or -1 but it is not necessarily equal to 1 or necessarily equal to -1. Consider the following examples:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b. False, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

c. True, we have:

$$\det(A) = \det(DBD^{-1}) = \det(D)\det(B)\det(D^{-1}) = \det(D)\det(B)\frac{1}{\det(D)} = \det(B)$$

d. False, consider $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the number of pivots in DBD^{-1} is 1 but the number of pivots in A is equal to 2. Thus, it is not possible to find D so that $A = DBD^{-1}$.

e. False, we have that A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$.

f. False, $\det(2A) = 2^3 \det(A) = 72$.

g. True, the rank of $R^T R$ is at most the rank of R (why?), i.e., it is at most 2. $R^T R$ is a 3×3 matrix with rank at most 2, so it is not invertible. Therefore, $\det(R^T R) = 0$.

h. False, consider $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

6. *Let f be a function with period 2π that satisfies $f(x) = x$ on $[-\pi, \pi)$. Find the Fourier series of f.*

Solution: We have to find coefficients a_i and b_i so that:

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + ...$$

We have, for $i \neq 0$:

$$a_i = \frac{\int_{-\pi}^{\pi} f(x)\cos(ix)\mathrm{d}x}{\int_{-\pi}^{\pi}\cos^2(ix)\mathrm{d}x} = \frac{\int_{-\pi}^{\pi} x\cos(ix)\mathrm{d}x}{\int_{-\pi}^{\pi}\frac{1-\cos(2ix)}{2}\mathrm{d}x} = \frac{\frac{x\sin(ix)}{i}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(ix)}{i}\mathrm{d}x}{\int_{-\pi}^{\pi}\frac{1-\cos(2ix)}{2}\mathrm{d}x} = 0$$

Also,

$$a_0 = \int_{-\pi}^{\pi} f(x)\mathrm{d}x = \int_{-\pi}^{\pi} x\mathrm{d}x = 0$$

In general, whenever we have an odd function($f(-x) = -f(x)$) then the a_i's are all zero. We have:

$$b_i = \frac{\int_{-\pi}^{\pi} f(x)\sin(ix)\mathrm{d}x}{\int_{-\pi}^{\pi} \sin^2(ix)\mathrm{d}x} = \frac{\int_{-\pi}^{\pi} x\sin(ix)\mathrm{d}x}{\int_{-\pi}^{\pi} \frac{1+\cos(2ix)}{2}\mathrm{d}x} = \frac{-\frac{x\cos(ix)}{i}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos(ix)}{i}\mathrm{d}x}{\int_{-\pi}^{\pi} \frac{1+\cos(2ix)}{2}\mathrm{d}x}$$

$$= \frac{-\frac{2\pi}{i}\cos(i\pi)}{\pi} = -\frac{2\cos(i\pi)}{i} = \frac{2}{i}(-1)^{i+1}$$

Note that $\int_{-\pi}^{\pi}\cos(ix)\mathrm{d}x = \int_{-\pi}^{\pi}\sin(ix)\mathrm{d}x = 0$.
In summary:

$$f(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x) + \frac{2}{5}\sin(5x) - \dots$$