Preparation problems for the discussion sections on November 4th and 6th

1. *Let* $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. *Find the least squares solution* $\widehat{\boldsymbol{x}}$ *of* $A\boldsymbol{x} = \boldsymbol{b}$.

2. *A scientist tries to find the relation between the mysterious quantities* x *and* y. *She measures the following values:*

x	1	2	3	4
y	2	5	9	17

(i) *Suppose that* y *is a linear function of the form* $a + bx$. *Set up the system of equations to find the coefficients* a *and* b.
(ii) *Find the best estimate for the coefficients.*
(iii) *Same question if we suppose that* y *is a quadratic function of the* $a + bx + cx^2$.

3. *The system of the equations* $A\boldsymbol{x} = \boldsymbol{b}$ *with*

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix},$$

is not consistent.

(i) *Find the least squares solution* $\widehat{\boldsymbol{x}}$ *for the equation* $A\boldsymbol{x} = \boldsymbol{b}$.
(ii) *Determine the least squares line for the data points* $(-1, 5), (0, 0), (1, 5), (2, 10)$.

4. *Let* $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *and* $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. *Using Gram-Schmidt, find an orthonormal basis for* $W = Span(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$, *using* $\boldsymbol{v}_1, \boldsymbol{v}_2,$ *and* $\boldsymbol{v}_3$.

5. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(i) *Calculate* $A^T A$. *What does this tell you about the columns of* A*?*
(ii) *Find an orthonormal basis* $\{q_1, q_2\}$ *for* $Col(A)$ *(starting with the columns of* A*!). Put* $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$. *What is* Q^{-1}*?*

6. *Let* $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. *Find the QR decomposition of* A*: write* $A = QR$ *where* Q *is a matrix with orthonormal columns and* R *is an upper triangular matrix.*

7. *Let*

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

the matrix for rotation over θ *(counter clockwise).*

(i) *Calculate* $Q_\theta^T Q_\theta$. *What does this tell you about the columns of* Q_θ*?*
(ii) *What is* Q_θ^{-1}*? Express* Q_θ^{-1} *in terms of another rotation matrix* Q_α.
(iii) *Show that if* $\boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ *then the vector* $\boldsymbol{x}$ *and the rotated vector* $Q_\theta \boldsymbol{x}$ *have the same length.*

8. *Let P be a permutation matrix, so each row and each column has a single non zero entry 1. Write* $P = \begin{bmatrix} P_1 & P_2 & \dots & P_n \end{bmatrix}$.

(i) *What is the dot product between the columns of P: what is $P_i \cdot P_j$?*

(ii) *What is P^{-1}?*