Preparation problems for the discussion sections on November 4th and 6th

1. *Let* $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. *Find the least squares solution* $\widehat{\boldsymbol{x}}$ *of* $A\boldsymbol{x} = \boldsymbol{b}$.

Solution: We first calculate $A^T A$ and $A^T \mathbf{b}$:

$$A^T A = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix},$$

$$A^T \mathbf{b} = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix} \widehat{\mathbf{x}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

It is easy to check that then

$$\widehat{\mathbf{x}} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{3} \end{bmatrix}.$$

2. *A scientist tries to find the relation between the mysterious quantities* x *and* y. *She measures the following values:*

x	1	2	3	4
y	2	5	9	17

(i) *Suppose that* y *is a linear function of the form* $a + bx$. *Set up the system of equations to find the coefficients* a *and* b.
(ii) *Find the best estimate for the coefficients.*
(iii) *Same question if we suppose that* y *is a quadratic function of the* $a + bx + cx^2$.

Solution: **a.** We set up the equation as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}.$$

b. We calculate

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}^T \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}$$

Now we solve

$$\left[\begin{array}{cc|c} 4 & 10 & 33 \\ 10 & 30 & 107 \end{array}\right] \xrightarrow{R2 \to R2 - 2.5R_1} \left[\begin{array}{cc|c} 4 & 10 & 33 \\ 0 & 5 & 24.5 \end{array}\right] \xrightarrow{R1 \to R1 - 2R2} \left[\begin{array}{cc|c} 4 & 0 & -16 \\ 0 & 5 & 24.5 \end{array}\right].$$

Hence $a = -4$ and $b = 4.9$.

c. We set up the equation as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}.$$

We calculate

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}^T \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \\ 375 \end{bmatrix}.$$

One can row reduce

$$\left[\begin{array}{ccc|c} 4 & 10 & 30 & 33 \\ 10 & 30 & 100 & 107 \\ 30 & 100 & 354 & 375 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.25 \\ 0 & 1 & 0 & -1.35 \\ 0 & 0 & 1 & 1.25 \end{array}\right].$$

So $a = 2.25$, $b = -1.35$ and $c = 1.25$.

3. *The system of the equations $A\boldsymbol{x} = \boldsymbol{b}$ with*

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix},$$

is not consistent.

(i) *Find the least squares solution $\widehat{\boldsymbol{x}}$ for the equation $A\boldsymbol{x} = \boldsymbol{b}$.*

(ii) *Determine the least squares line for the data points $(-1, 5), (0, 0), (1, 5), (2, 10)$.*

Solution:

(i) We first calculate $A^T A$ and $A^T\mathbf{b}$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \widehat{\mathbf{x}} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

We have

$$\left[\begin{array}{cc|c} 4 & 2 & 20 \\ 2 & 6 & 20 \end{array}\right] \xrightarrow{R2 \to R2 - 1/2 R1} \left[\begin{array}{cc|c} 4 & 2 & 20 \\ 0 & 5 & 10 \end{array}\right].$$

Hence,

$$\widehat{\mathbf{x}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(ii) Denoting the least squares line as $y = ax + b$, we have to find $\begin{bmatrix} b \\ a \end{bmatrix}$ so that $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$ is the closest possible value to $\begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$. From the first part, $\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Hence, the least squares line is $y = ax + b = 2x + 4$.

4. *Let* $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *and* $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. *Using Gram-Schmidt, find an orthonormal basis for* $W = Span(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$, *using* $\boldsymbol{v}_1, \boldsymbol{v}_2,$ *and* $\boldsymbol{v}_3$.

Solution: Set

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\| \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Then

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1}{\|\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}{\| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \|} = \frac{\begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}}{\| \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \|} = \sqrt{\frac{3}{2}} \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Finally,

$$\mathbf{u}_3 = \frac{\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2}{\|\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2\|}$$

$$= \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} - (\begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}) \begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}}{\| \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - (\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} - (\begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}) \begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \|}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Now $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of W.

5. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(i) *Calculate $A^T A$. What does this tell you about the columns of A?*
(ii) *Find an orthonormal basis $\{q_1, q_2\}$ for Col(A) (starting with the columns of A!). Put $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$. What is Q^{-1}?*

Solution:

(i) We have:

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since only entries on the main diagonal are nonzero, columns of A are orthogonal to each other.

(ii) Since we already know that columns of A are orthogonal to each other, to find an orthonormal basis for Col(A) it is enough to divide each column by its length. Hence: (note that for non-zero vectors, orthogonality implies linear independence)

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Q is an orthogonal matrix, so:

$$Q^{-1} = Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

6. *Let* $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. *Find the QR decomposition of A: write $A = QR$ where Q is a matrix with orthonormal columns and R is an upper triangular matrix.*

Solution: Let W be the column space of A. Then $W = \text{span}(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix})$. By applying the Gram-Schmidt process to these vectors, we have:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix}1\\0\\1\end{bmatrix}}{\|\begin{bmatrix}1\\0\\1\end{bmatrix}\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

Then

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1}{\|\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\|} = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}}{\|\begin{bmatrix}1\\0\\0\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}\|} = \frac{\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}}{\|\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}.$$

Finally,

$$\begin{aligned}
\mathbf{u}_3 &= \frac{\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2}{\|\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2\|} \\
&= \frac{\begin{bmatrix}2\\1\\0\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}}{\|\begin{bmatrix}2\\1\\0\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} - (\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}\|} \\
&= \begin{bmatrix}0\\1\\0\end{bmatrix}.
\end{aligned}$$

Therefore, by using the Gram-Schmidt process we get the following orthonormal basis for $W = \text{Col}(A)$:

$$\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\}$$

Then set

$$Q = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}$$

Now we determine R. We have:

$$R = Q^T A = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}\sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2}\\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2}\\ 0 & 0 & 1\end{bmatrix}.$$

Check that $A = QR$.

[Note that it is not really necessary to compute this matrix product to find R. Can you see how all entries of R have occured as inner products during Gram-Schmidt?]

7. *Let*

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

the matrix for rotation over θ (counter clockwise).

(i) *Calculate $Q_\theta^T Q_\theta$. What does this tell you about the columns of Q_θ?*
(ii) *What is Q_θ^{-1}? Express Q_θ^{-1} in terms of another rotation matrix Q_α.*
(iii) *Show that if $\boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ then the vector $\boldsymbol{x}$ and the rotated vector $Q_\theta \boldsymbol{x}$ have the same length.*

Solution:

(i) We have:

$$Q_\theta^T Q_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that the columns of Q_θ form an orthonormal basis for $\mathbb{R}^2$.

(ii) By the first part, we have $Q_\theta^{-1} = Q_\theta^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. It is easy to see that the inverse of the rotation by θ is the rotation by $-\theta$, therefore:

$$Q_\theta^{-1} = Q_{-\theta}$$

(iii) We have:

$$Q_\theta \mathbf{x} = Q_\theta \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$$

Thus,

$$\begin{aligned} \text{length}(Q_\theta \mathbf{x}) =& \sqrt{(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2} = \sqrt{a^2(\cos^2\theta + \sin^2\theta) + b^2(\cos^2\theta + \sin^2\theta)} \\ =& \sqrt{a^2 + b^2} = \text{length}(\mathbf{x}) \end{aligned}$$

8. *Let P be a permutation matrix, so each row and each column has a single non zero entry 1. Write $P = \begin{bmatrix} P_1 & P_2 & \dots & P_n \end{bmatrix}$.*

(i) *What is the dot product between the columns of P: what is $P_i \cdot P_j$?*
(ii) *What is P^{-1}?*

Solution:

(i) We have $P_i \cdot P_j = 0$ if $i \neq j$ and $P_i \cdot P_i = 1$. This is because the columns of P are the standard basis vectors of $\mathbb{R}^n$ in a different order.
(ii) From the first part, we know that columns of P form an orthonormal basis, i.e., P is orthogonal. Hence, we have:

$$P^{-1} = P^T$$