Preparation problems for the discussion sections on October 28th and 30th

1. *Let* $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. *Let* $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. *Can you find real numbers* c_1, c_2 *such that*

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2?$$

Solution: Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal (i.e. $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$), we have that if

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

for some real number c_1, c_2, then

$$\mathbf{v} \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1$$

and

$$\mathbf{v} \cdot \mathbf{u}_2 = c_1\mathbf{u}_1 \cdot \mathbf{u}_2 + c_2\mathbf{u}_2 \cdot \mathbf{u}_2 = c_2\mathbf{u}_2 \cdot \mathbf{u}_2.$$

Hence

$$c_1 = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} = -\frac{3}{5}$$

and

$$c_2 = \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} = \frac{6}{9} = \frac{2}{3}.$$

However, we see that $-\frac{3}{5}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 \neq \mathbf{v}$, so it is not possible to find real numbers c_1, c_2 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$.

The numbers that we found, however, are "best possible" in the sense that the two sides are as close as possible. In other words, $-\frac{3}{5}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2$ is the orthogonal projection of $\mathbf{v}$ onto the space spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

[Note that you can solve this problem in many other ways. The way above serves to make us more familiar with notions such as orthogonal projections.]

2. *Let* $W = \mathrm{Span}\{\boldsymbol{v}\}$, *where* $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, *be a subspace of* $\mathbb{R}^3$. *Find the projections* $\boldsymbol{a}_W, \boldsymbol{b}_W, \boldsymbol{c}_W$ *of the vectors*

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

onto the subspace W. *Interpret your results geometrically.*

Solution: We have,

$$\mathbf{a}_W = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix},$$

$$\mathbf{b}_W = \frac{\mathbf{b}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\begin{bmatrix}2\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{0}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\mathbf{c}_W = \frac{\mathbf{c}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\begin{bmatrix}2\\2\\2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

The fact that $\mathbf{b}_W$ is zero means that $\mathbf{b}$ is orthogonal to W. In this, and the other two cases, we obtain the vector in W which is closest to the vector that we start with.

3. *Let* $W = \text{Span}\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\}$ *be a subspace of* $\mathbb{R}^4$.

(i) *Find the closest point to* $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ *on the subspace* W.

(ii) *Find the projection matrix,* P*, corresponding to the projection onto* W.

(iii) *Use the projection matrix,* P*, to find the projection of* $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ *onto the subspace* W.

Solution:

(i) The closest point is the orthogonal projection:

$$\frac{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

(ii) The projections of the four standard basis vectors are

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{3}{4}\\-\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

Hence, the projection matrix is:

$$P = \begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}$$

(iii) Using P, we find that the orthogonal projection is

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_W = \begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

4. *Let* $W = \text{Span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\}$ *and* $V = \text{Span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}\}$ *be subspaces of* $\mathbb{R}^3$.

(i) *Find the projection matrices, P and Q, corresponding to the projections onto W and V, respectively.*

(ii) *Check that $PQ = QP$. Can you interpret PQ as a projection matrix?*

Solution:

(i) The projections onto W of the three standard basis vectors are

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{5}{6}\\-\frac{1}{6}\\\frac{1}{3}\end{bmatrix},$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{6}\\\frac{5}{6}\\\frac{1}{3}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}.$$

Hence, the projection matrix corresponding to the orthogonal projection onto W is:

$$P = \begin{bmatrix}\frac{5}{6} & -\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

On the other hand, the projections onto V of the three standard basis vectors are

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_V = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}_V = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_V = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Hence, the projection matrix corresponding to the orthogonal projection onto V is:

$$P = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

(ii) $PQ = QP$ is the matrix corresponding to the orthogonal projection onto the intersection of W and V (the space of all vectors in both W and V), that is $W \cap V = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$.

[Note: since $\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix} = 0$ if you compute orthogonal projection onto W and then onto V the answer will be same as computing orthogonal projection onto V and then onto W]

5. *Let* $A = \begin{bmatrix}1 & -1\\ 1 & 1\\ 0 & 0\end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$.

a. *Does* $\boldsymbol{b}$ *belong to the column space of* A*? Can you solve* $A\boldsymbol{x} = \boldsymbol{b}$*?*

b. *What do you expect the projection of* $\boldsymbol{b}$ *onto* $W = Col(A)$ *to be?*

c. *Find the projection* $\widehat{\boldsymbol{b}}$ *of* $\boldsymbol{b}$ *onto* $Col(A)$*, and then solve* $A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$*. (The vector* $\widehat{\boldsymbol{x}}$ *is called* the least square solution *of* $A\boldsymbol{x} = \boldsymbol{b}$*.)*

d. *Solve the equation* $A^T A\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$*. Compare with your result of the previous part! (This equation is called the* normal equation *of* $A\boldsymbol{x} = \boldsymbol{b}$*.)*

e. *Answer these questions for* A *as above but with* $\boldsymbol{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ *(and then* $\boldsymbol{b} = \begin{bmatrix}0\\0\\4\end{bmatrix}$*).*

Solution: **a.** No, **b** does not belong to the column space of A, because it is not a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. Hence there is no solution to $A\mathbf{x} = \mathbf{b}$.

b. W is the span of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. It can easily be seen that W is the set of vectors in $\mathbb{R}^3$

whose third entry is 0. Hence $\begin{bmatrix}4\\5\\0\end{bmatrix}$ is in W. Note

$$\begin{bmatrix}0\\0\\6\end{bmatrix} = \mathbf{b} - \begin{bmatrix}4\\5\\0\end{bmatrix}$$

is orthogonal to W. Hence $\begin{bmatrix}4\\5\\0\end{bmatrix}$ should be the orthogonal projection of $\mathbf{b}$ onto W.

[Note that the columns of A are an orthogonal basis for W. Hence, we actually know how to compute the orthogonal projection of $\mathbf{b}$ onto W. The result will be as above, and the computation is given in Step 1 below.]

c. Step 1: Find the orthogonal projection of $\mathbf{b}$ onto W.

$$\widehat{\mathbf{b}} = \frac{\begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}} \begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix}}{\begin{bmatrix}-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix}} \begin{bmatrix}-1\\1\\0\end{bmatrix} = 4.5 \begin{bmatrix}1\\1\\0\end{bmatrix} + .5 \begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}4\\5\\0\end{bmatrix}.$$

Step 2: Solve $A\widehat{\mathbf{x}} = \widehat{\mathbf{b}}$.

$$\left[\begin{array}{cc|c}1 & -1 & 4\\1 & 1 & 5\\0 & 0 & 0\end{array}\right] \xrightarrow{R2 \to R2 - R_1} \left[\begin{array}{cc|c}1 & -1 & 4\\0 & 2 & 1\\0 & 0 & 0\end{array}\right] \xrightarrow{R1 \to R1 + .5R2} \left[\begin{array}{cc|c}1 & 0 & 4.5\\0 & 2 & 1\\0 & 0 & 0\end{array}\right].$$

Hence,

$$\widehat{\mathbf{x}} = \begin{bmatrix}4.5\\.5\end{bmatrix}.$$

[Note that this was unnecessary! When projecting $\mathbf{b}$ onto W, we already expressed the result as a linear combination of the columns of A.]

d. We first calculate $A^T A$ and $A^T\mathbf{b}$:

$$A^T A = \begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\end{bmatrix} \begin{bmatrix}1 & -1\\1 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}9\\1\end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} \widehat{\mathbf{x}} = \begin{bmatrix}9\\1\end{bmatrix}.$$

Clearly, then

$$\widehat{\mathbf{x}} = \begin{bmatrix}4.5\\.5\end{bmatrix}.$$

e. I leave the details to you, but here are the solutions. For $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$:

$$\widehat{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \widehat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note that in this case $\widehat{\mathbf{b}} = \mathbf{b}$ and $\widehat{\mathbf{x}}$ is a solution (not just a least squares solution) of $A\mathbf{x} = \mathbf{b}$.

For $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$:

$$\widehat{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \widehat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This was to be expected because $\mathbf{b}$ is orthogonal to the columns of A.