Preparation problems for the discussion sections on October 14th and 16th

1. *Let* $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Find the length of* $\boldsymbol{v}$. *Find a vector* $\boldsymbol{u}$ *in the direction of* $\boldsymbol{v}$ *that has length* 1 . *Find a vector* $\boldsymbol{w}$ *that is orthogonal to* $\boldsymbol{v}$.

2. *Let* $\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, *and* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *Find real numbers* c_1, c_2 *such that*

$$\boldsymbol{v} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2.$$

3. *Let* $V = \{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + b + c + d = 0\}$ *be a subspace of* $\mathbb{R}^4$.

(a) *Find a basis for* V.
(b) *Find a vector that is orthogonal to* V.
(c) *Can you find two linearly independent vectors that are orthogonal to* V*?*

4. *Let* $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(a) *Find an echelon form* U *of* A. *What are the column spaces* $Col(A)$, $Col(U)$*? Are they equal?*
(b) *Find a basis for* $Col(U)$ *and a basis for* $Col(A)$.
(c) *What are the row spaces* $Col(A^T)$, *and* $Col(U^T)$. *Are they equal?*
(d) *Find a basis for the row space of* A, $Col(A^T)$.

5. *Let* $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

(a) *Find a basis for* $Nul(B)$.
(b) *Find two linear independent vectors that are orthogonal to* $Nul(B)$.
(c) *Is there a non-zero vector in* $\mathbb{R}^2$ *orthogonal to* $Col(B)$*?*

6. *Let* $\mathcal{B} := \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ *be a basis of* $\mathbb{R}^3$. *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be the linear transformation that maps* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in* $\mathbb{R}^3$ *to* $\begin{bmatrix} z \\ x \\ y \end{bmatrix}$. *Determine the matrix corresponding to* T *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{B}$.

7. *Let* $I\colon \mathbb{P}^3 \to \mathbb{P}^4$ *be the linear transformation that maps* $p(t)$ *to*

$$tp(t) + p'(t)$$

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ *of* $\mathbb{P}^3$ *and the basis* $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ *of* $\mathbb{P}^4$*. Determine the matrix which represents* I *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{C}$*.*

8. *True or False? Justify your answers.*

(a) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}$ *given by* $T\begin{bmatrix} a \\ b \end{bmatrix} = \sqrt{a^2 + b^2}$ *is a linear transformation.*

(b) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$ *is a linear transformation.*

(c) *If* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in* $\mathbb{R}^2$ *are such that* $\boldsymbol{u}.\boldsymbol{v} = 0$ *(*$\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are orthogonal) then* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are perpendicular (geometrically) to each other.*

(d) *Let* V *be a subspace and* $\boldsymbol{u}, \boldsymbol{v}$ *be two vectors in* V*, then* $\boldsymbol{v} - \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{u}.\boldsymbol{u}}\boldsymbol{u}$ *is orthogonal to* $\boldsymbol{u}$*.*

(e) *Let* $T : V \to W$ *be a linear transformation and* $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ *be vectors in* V*. If* $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_n)$ *are linearly independent then* $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ *are also linearly independent.*

(f) *Let* $T : V \to W$ *be a linear transformation and* $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ *be vectors in* V*. If* $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ *are linearly independent then* $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_n)$ *are also linearly independent.*

(g) *Let* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation. The dimension of the image of* T *is equal to* 2*.*