Preparation problems for the discussion sections on October 7th and 9th

1. *Determine a basis for each of the following subspaces:*

(i) $H = \{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \in \mathbb{R}\}$,

(ii) $K = \{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 3b + c = 0\}$,

(iii) $Col(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix})$,

(iv) $Nul(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix})$.

2. *Determine the dimension of $Nul(A)$ and $Col(A)$, where*

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

3. *Let A, B be two 4×3 matrices. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the columns of A and let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the columns of B.*

(i) *Suppose that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is linearly independent. Find a basis for $Col(A)$ and describe $Nul(A)$.*

(ii) *Suppose that $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ is linearly independent and $\boldsymbol{b}_3 = 2\boldsymbol{b}_1 + 7\boldsymbol{b}_2$. Find a basis for $Col(B)$ and a basis for $Nul(B)$.*

4. *Let $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.*

(i) *Let $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Express $\boldsymbol{v}$ in terms of the basis $\mathcal{B}$.*

(ii) *Let $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Express $\boldsymbol{w}$ in terms of the basis $\mathcal{B}$.*

(iii) *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined such that $T(\boldsymbol{v})$ is expressing $\boldsymbol{v}$ in terms of the basis $\mathcal{B}$. (Convince yourself that this is a linear transformation.) Determine the matrix that represents T with respect to the standard basis of $\mathbb{R}^2$.*

5. *Let* $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation such that*

$$L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\8\\4\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\1\end{bmatrix}.$$

What is $L\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$*?*

6. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *be the linear transformations with*

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\0\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

(i) *Consider the basis* $\mathcal{B}_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{B}_2 = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ *of* $\mathbb{R}^3$*. Determine the matrix* A *which represents* T *with respect to the bases* $\mathcal{B}_1$ *and* $\mathcal{B}_2$*. Do you have* $T(\boldsymbol{x}) = A\boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathbb{R}^2$*?*

(ii) *Consider the basis* $\mathcal{C}_1 := \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{C}_2 = \{\begin{bmatrix}5\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ *of* $\mathbb{R}^3$*. Determine the matrix* B *which represents* T *with respect to the bases* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. Do you have* $T(\boldsymbol{x}) = B\boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathbb{R}^2$*?*

7. *Let* $I\colon \mathbb{P}^3 \to \mathbb{P}^4$ *be the integration linear transformation that maps* p *to*

$$\int_0^t p(t)dt.$$

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ *of* $\mathbb{P}^3$ *and the basis* $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ *of* $\mathbb{P}^4$*. Determine the matrix which represents* I *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{C}$*.*