Preparation problems for the discussion sections on September 16th and 18th

1. (1) *Find a matrix E such that:*

$$E\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - 2R_1 \\ R_3 \end{bmatrix}$$

Which matrix E^{-1} undoes the row operation implemented by E? What is $E^{-1}E$?

(2) *Find a matrix F such that:*

$$F\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix}$$

Which matrix F^{-1} undoes the row operation implemented by F? What is $F^{-1}F$?

(3) *Find a matrix G such that:*

$$G\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 3R_2 \\ R_3 \end{bmatrix}$$

Which matrix G^{-1} undoes the row operation implemented by G? What is $G^{-1}G$?

2. *Consider the matrix:*

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

Decompose the matrix A into LU, where L is a lower triangular matrix and U is an upper triangular matrix. Then use this factorization to solve:

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

That means, find a vector $\mathbf{c}$ in $\mathbb{R}^3$ such that:

$$L\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

and then find a vector $\boldsymbol{x}$ in $\mathbb{R}^3$ such that:

$$U\boldsymbol{x} = \mathbf{c}$$

3. *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.

(1) *Show that $A = LU$.*

(2) *Let A_i be the $i \times i$ matrix introduced by the first i rows and the first i columns of A, for $i = 1, 2, 3$. What is an LU decomposition of A_i, for $i = 1, 2, 3$?*

4. *(more challenging) Let A and B be $n \times n$ matrices such that $AB = I$.*

(1) *What is the reduced echelon form of A?*

(2) *Show that $BA = I$.*

5. *Answer the following true-false questions. Explain your answer.*

(1) *If A is invertible then $A\boldsymbol{x} = 0$ has exactly one solution, $\boldsymbol{x} = 0$.*

(2) *If A is invertible then AB is also invertible.*

(3) *If A and B are invertible then $A + B$ is also invertible.*

(4) *If A is invertible then the reduced echelon form of A is equal to I.*

6. *If* $G = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, *find G^{-1}. Check that $G^{-1}G = I$.*

7. *Let* $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. *Use the Gauss-Jordan method to either find the inverse of A or to show that A is not invertible.*

8. *Calculate the inverse of the matrix:*

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

9. *Consider the equation:*

$$-\frac{d^2u}{dx^2} = 4\pi^2 \sin 2\pi x, \quad u(0) = u(1) = 0$$

(1) *Write down the 3 by 3 matrix equation with $h = \frac{1}{4}$.*

(2) *Solve for u_1, u_2, u_3 and find their error in comparison with the true solution $u = \sin 2\pi x$ at $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$.*