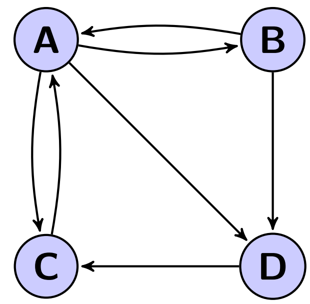
Review

- We model a surfer randomly clicking webpages.

Let $\mathrm{PR}_n(A)$ be the probability that he is at A (after n steps).

$\mathrm{PR}_n(A) = \mathrm{PR}_{n-1}(B) \cdot \frac{1}{2} + \mathrm{PR}_{n-1}(C) \cdot \frac{1}{1} + \mathrm{PR}_{n-1}(D) \cdot \frac{0}{1}$

$$\begin{bmatrix} \mathrm{PR}_n(A) \\ \mathrm{PR}_n(B) \\ \mathrm{PR}_n(C) \\ \mathrm{PR}_n(D) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}}_{=T} \begin{bmatrix} \mathrm{PR}_{n-1}(A) \\ \mathrm{PR}_{n-1}(B) \\ \mathrm{PR}_{n-1}(C) \\ \mathrm{PR}_{n-1}(D) \end{bmatrix}$$



- The transition matrix T is a **Markov matrix**.

  Its columns add to 1 and it has no negative entries.

- The **Page rank** of page A is $\mathrm{PR}(A) = \mathrm{PR}_\infty(A)$. (assuming the limit exists)

  It is the probability that the surfer is at page A after n steps (with $n \to \infty$).

- The **PageRank vector** $\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix}$ satisfies $\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = T \begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix}$.

  It is an eigenvector of the transition matrix T with eigenvalue 1.

- $$\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  $\Longrightarrow$ eigenspace of $\lambda = 1$ spanned by $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$

  $\Longrightarrow$ $\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \frac{3}{16} \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$ This the PageRank vector.

- The corresponding ranking of the webpages is A, C, D, B.

Remark 1. In practical situations, the system might be too large for finding the eigenvector by elimination.

- Google reports having met about 60 trillion webpages

  Google's search index is over 100,000,000 gigabytes

  Number of Google's servers secret; about 2,500,000

  More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)

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- Thus we have a gigantic but very sparse matrix.

An alternative to elimination is the **power method**:

If T is an (acyclic and irreducible) Markov matrix, then for any $\boldsymbol{v}_0$ the vectors $T^n\boldsymbol{v}_0$ converge to an eigenvector with eigenvalue 1.

Here: $T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$$

$$T\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/12 \\ 1/3 \\ 5/24 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}$$

Note that the ranking of the webpages is already A, C, D, B if we stop here.

$$T^2\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix}$$

$$T^3\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.396 \\ 0.125 \\ 0.292 \\ 0.188 \end{bmatrix}$$

Remark 2.

- If all entries of T are positive, then the power method is guaranteed to work.
- In the context of PageRank, we can make sure that this is the case, by replacing T with

$$(1-p)\cdot\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} + p\cdot\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

  Just to make sure: still a Markov matrix, now with positive entries

  Google used to use $p = 0.15$.

- Why does $T^n\boldsymbol{v}_0$ converge to an eigenvector with eigenvalue 1?

  Under the assumptions on T, its other eigenvalues λ satisfy $|\lambda| < 1$.

  Now, think in terms of a basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of eigenvectors:

  $T^m(c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}_n) = c_1\lambda_1^m\boldsymbol{x}_1 + \ldots + c_n\lambda_n^m\boldsymbol{x}_n$

  As m increases, the terms with λ_i^m for $\lambda_i \neq 1$ go to zero, and what is left over is an eigenvector with eigenvalue 1.

Linear differential equations

Example 3. Which functions $y(t)$ satisfy the **differential equation** $y' = y$?

Solution: $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

Recall from Calculus the Taylor series $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

Example 4. The differential equation $y' = ay$ with **initial condition** $y(0) = C$ is solved by $y(t) = Ce^{at}$. (This solution is unique.)

Why? Because $y'(t) = aCe^{at} = ay(t)$ and $y(0) = C$.

Example 5. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Key idea: to solve $\boldsymbol{y}' = A\boldsymbol{y}$, introduce e^{At}

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Review of diagonalization

- If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ.
- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Let A be $n \times n$ with independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$.

- the columns of P are the eigenvectors
- the diagonal matrix D has the eigenvalues on the diagonal

Example 6. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Solution.

- A has eigenvalues 2 and 4. (We did that in an earlier class!)
  - $\lambda = 2$: $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \Longrightarrow$ eigenspace $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$
  - $\lambda = 4$: $\begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \Longrightarrow$ eigenspace $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$
- $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$
- $A = PDP^{-1}$

  For many applications, it is not needed to compute P^{-1} explicitly.
- We can check this by verifying $AP = PD$:

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$$