Review

- **Eigenvector** equation: $A\boldsymbol{x} = \lambda\boldsymbol{x} \iff (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

  λ is an **eigenvalue** of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0.$

- An $n \times n$ matrix A has up to n different eigenvalues λ.
  - The **eigenspace** of λ is $\mathrm{Nul}(A - \lambda I)$.

    That is, all eigenvectors of A with eigenvalue λ.
  - If λ has **multiplicity** m, then A has up to m eigenvectors for λ.

    At least one eigenvector is guaranteed (because $\det(A - \lambda I) = 0$).

- Test yourself! What are the eigenvalues and eigenvectors?
  - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1, 1$ (ie. multiplicity 2), eigenspace is $\mathbb{R}^2$
  - $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\lambda = 0, 0$, eigenspace is $\mathbb{R}^2$
  - $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\lambda = 2, 2$, eigenspace is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Diagonalization

Diagonal matrices are very easy to work with.

Example 1. For instance, it is easy to compute their powers.

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 2^2 & & \\ & 3^2 & \\ & & 4^2 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} 2^{100} & & \\ & 3^{100} & \\ & & 4^{100} \end{bmatrix}$

Example 2. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$

Solution.

- Characteristic polynomial: $\begin{vmatrix} 6 - \lambda & -1 \\ 2 & 3 - \lambda \end{vmatrix} = \ldots = (\lambda - 4)(\lambda - 5)$
  - $\lambda_1 = 4$: $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0} \implies$ eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  - $\lambda_2 = 5$: $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \implies$ eigenvector $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Key observation: $A^{100}\boldsymbol{v}_1 = \lambda_1^{100}\boldsymbol{v}_1$ and $A^{100}\boldsymbol{v}_2 = \lambda_2^{100}\boldsymbol{v}_2$

  For A^{100}, we need $A^{100}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^{100}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  $\Longrightarrow \quad A^{100}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^{100}\left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -4^{100}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \cdot 5^{100}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  $\Longrightarrow \quad A^{100} = \begin{bmatrix} 2 \cdot 5^{100} - 4^{100} & * \\ 2 \cdot 5^{100} - 2 \cdot 4^{100} & * \end{bmatrix}$
- We find the second column of A^{100} likewise. Left as exercise!

The key idea of the previous example was to work with respect to a basis given by the eigenvectors.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$.

- the columns of P are the eigenvectors
- the diagonal matrix D has the eigenvalues on the diagonal

Such a diagonalization is possible if and only if A has enough eigenvectors.

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Example 3.

  

Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

By the way: "not a universal law but only a fascinatingly prevalent tendency" — Coxeter

Did you notice: $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, ...

The **golden ratio** $\varphi = 1.618\ldots$ Where's that coming from?

By the way, this φ is the *most irrational* number (in a precise sense).

- $F_{n+1} = F_n + F_{n-1} \quad \Longrightarrow \quad \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
- Hence: $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$ $\qquad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- But we know how to compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$!

Solution. (Exercise to fill in all details!)

- The characteristic polynomial of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2 - \lambda - 1$.
- The eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden mean!) and $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$.
- Corresponding eigenvectors: $\boldsymbol{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$
- Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \boldsymbol{v}_1 + c_1 \boldsymbol{v}_2$. $\qquad (c_1 = \frac{1}{\sqrt{5}},\ c_2 = -\frac{1}{\sqrt{5}})$
- $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^n c_1 \boldsymbol{v}_1 + \lambda_2^n c_2 \boldsymbol{v}_2$
- Hence, $F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$.

  That's **Binet's formula**.
- But $|\lambda_2| < 1$, and so $F_n \approx \lambda_1^n c_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$.

  In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$. Don't you feel powerful!?

Practice problems

Problem 1. Find, if possible, the diagonalization of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

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