Review

- Let A be an $m\times n$ matrix of rank n. (columns independent)

  Then we have the **QR decomposition** $A = QR$,

  - where Q is $m\times n$ with orthonormal columns, and
  - R is upper triangular and invertible.

- To obtain

$$\begin{bmatrix} | & | & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} | & | & \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^T\boldsymbol{a}_1 & \boldsymbol{q}_1^T\boldsymbol{a}_2 & \boldsymbol{q}_1^T\boldsymbol{a}_3 & \cdots \\ & \boldsymbol{q}_2^T\boldsymbol{a}_2 & \boldsymbol{q}_2^T\boldsymbol{a}_3 & \\ & & \boldsymbol{q}_3^T\boldsymbol{a}_3 & \\ & & & \ddots \end{bmatrix}$$

  - Gram–Schmidt on (columns of) A, to get (columns of) Q.
  - Then, $R = Q^T A$. (actually unnecessary!)

Example 1. The QR decomposition is also used to solve systems of linear equations. (we assume A is $n\times n$, and A^{-1} exists)

$$\begin{aligned} A\boldsymbol{x} = \boldsymbol{b} \quad &\Longleftrightarrow \quad QR\boldsymbol{x} = \boldsymbol{b} \\ &\Longleftrightarrow \quad R\boldsymbol{x} = Q^T\boldsymbol{b} \end{aligned}$$

The last system is triangular and is solved by back substitution.

QR is a little slower than LU but makes up in numerical stability.

If A is not $n\times n$ and invertible, then $R\boldsymbol{x} = Q^T\boldsymbol{b}$ gives the least squares solutions!

Example 2. The QR decomposition is very useful for solving least squares problems:

$$\begin{aligned} A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b} \quad &\Longleftrightarrow \quad \underbrace{(QR)^TQR}_{=R^TQ^TQR}\hat{\boldsymbol{x}} = (QR)^T\boldsymbol{b} \\ &\Longleftrightarrow \quad R^TR\hat{\boldsymbol{x}} = R^TQ^T\boldsymbol{b} \\ &\Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b} \end{aligned}$$

Again, the last system is triangular and is solved by back substitution.

> $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
> $\Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$ (where $A = QR$)

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Application: Fourier series

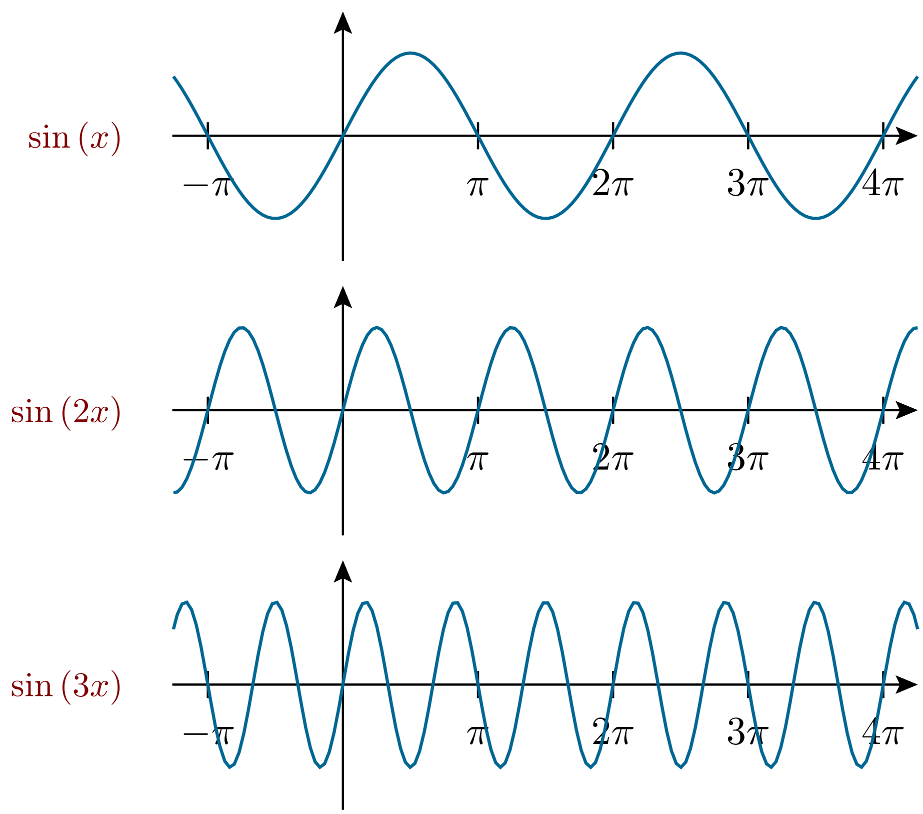
Review. Given an orthogonal basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$, we express a vector $\boldsymbol{x}$ as:

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots, \qquad \underbrace{c_i\boldsymbol{v}_i = \frac{\langle \boldsymbol{x}, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle}\boldsymbol{v}_i}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_i}$$

A **Fourier series** of a function $f(x)$ is an infinite expansion:

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

Example 3.



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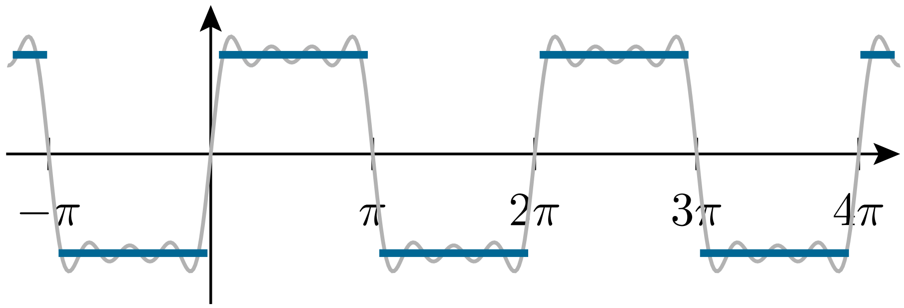
Example 4. (just a preview)

$$\text{blue function} = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \ldots\right)$$



- We are working in the vector space of functions $\mathbb{R} \to \mathbb{R}$.
  - More precisely, "nice" (say, piecewise continuous) functions that have period 2π.
  - These are infinite dimensional vector spaces.
- The functions

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$$

are a basis of this space. In fact, an **orthogonal basis**!

That's the reason for the success of Fourier series.

But what is the inner product on the space of functions?

- Vectors in $\mathbb{R}^n$: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = v_1 w_1 + \ldots + v_n w_n$
- Functions: $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \mathrm{d}x$

Why these limits? Because our functions have period 2π.

Example 5. Show that $\cos(x)$ and $\sin(x)$ are orthogonal.

Solution.

$$\langle \cos(x), \sin(x) \rangle = \int_0^{2\pi} \cos(x) \sin(x) \mathrm{d}x = \left[\frac{1}{2}(\sin(x))^2\right]_0^{2\pi} = 0$$

More generally, $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$ are all orthogonal to each other.

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Example 6. What is the norm of $\cos(x)$?

Solution.

$$\langle \cos(x), \cos(x) \rangle = \int_0^{2\pi} \cos(x)\cos(x)\mathrm{d}x = \pi$$

Why? There's many ways to evaluate this integral. For instance:

- you could use integration by parts,
- you could use a trig identity,
- here's a simple way:
  - $\int_0^{2\pi} \cos^2(x)\mathrm{d}x = \int_0^{2\pi} \sin^2(x)\mathrm{d}x$ ($\cos$ and $\sin$ are just a shift apart)
  - $\cos^2(x) + \sin^2(x) = 1$
  - So: $\int_0^{2\pi} \cos^2(x)\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}x = \pi$

Hence, $\cos(x)$ is not normalized. It has norm $\|\cos(x)\| = \sqrt{\pi}$.

Example 7. The same calculation shows that $\cos(kx)$ and $\sin(kx)$ have norm $\sqrt{\pi}$ as well.

> Fourier series of $f(x)$:
>
> $$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

Example 8. How do we find a_1?

Or: how much cosine is in a function $f(x)$?

Solution.

$$a_1 = \frac{\langle f(x), \cos(x) \rangle}{\langle \cos(x), \cos(x) \rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(x)\mathrm{d}x$$

> $f(x)$ has the Fourier series
>
> $$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$
>
> where
>
> $$a_k = \frac{\langle f(x), \cos(kx) \rangle}{\langle \cos(kx), \cos(kx) \rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\mathrm{d}x,$$
> $$b_k = \frac{\langle f(x), \sin(kx) \rangle}{\langle \sin(kx), \sin(kx) \rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\mathrm{d}x,$$
> $$a_0 = \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x.$$

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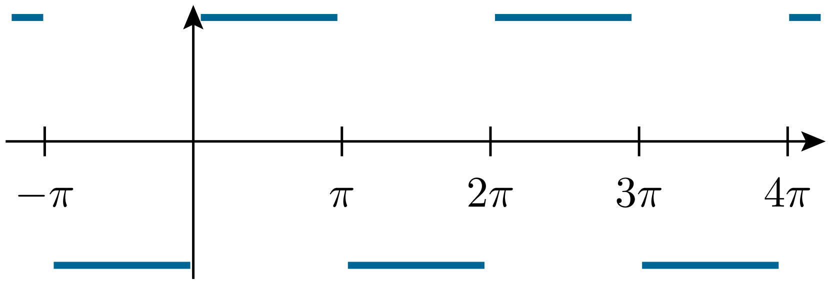
Example 9. Find the Fourier series of the 2π-periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0), \\ +1, & \text{for } x \in (0, \pi). \end{cases}$$

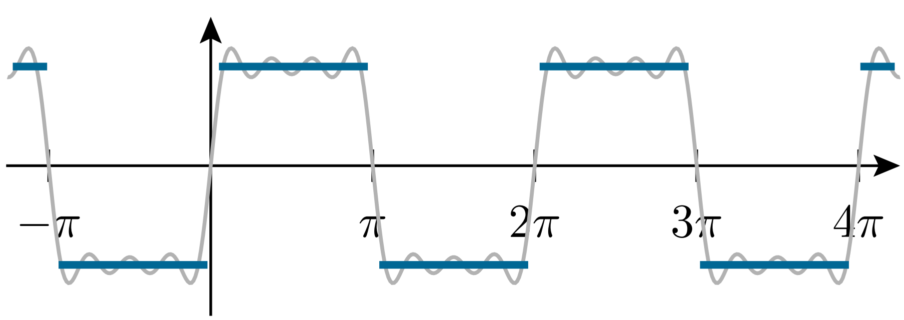


Solution. Note that $\int_0^{2\pi}$ and $\int_{-\pi}^{\pi}$ are the same here. (why?!)

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos(nx)\mathrm{d}x + \int_0^{\pi} \cos(nx)\mathrm{d}x\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \sin(nx)\mathrm{d}x + \int_0^{\pi} \sin(nx)\mathrm{d}x\right] \\ &= \frac{2}{\pi}\left[\int_0^{\pi} \sin(nx)\mathrm{d}x\right] \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\ &= \frac{2}{\pi n}[1 - \cos(n\pi)] \\ &= \frac{2}{\pi n}[1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

In conclusion,

$$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \ldots\right).$$



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