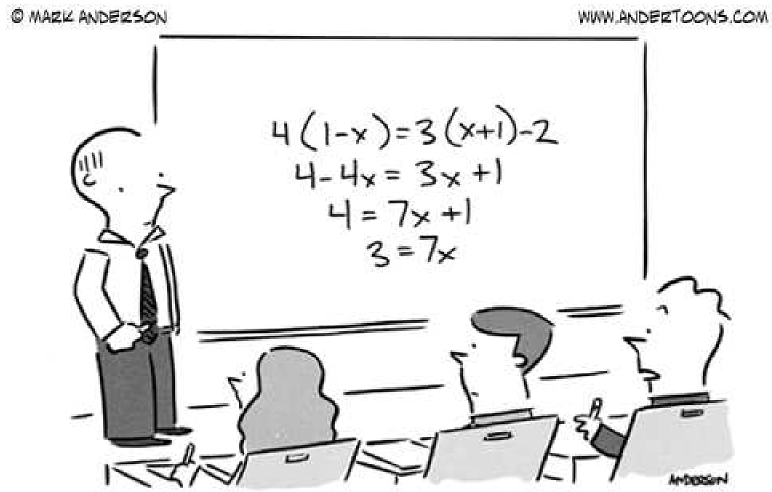

"Wouldn't it be more efficient to just find who's complicating equations and ask them to stop?"

- We can deal with "complicated" linear systems, but what to do if there is no solutions and we want a "best" approximate solution?

  This is important for many applications, including fitting data.

- Suppose $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. This means $\boldsymbol{b}$ is not in $\mathrm{Col}(A)$.

  Idea: find "best" approximate solution by replacing $\boldsymbol{b}$ with its projection onto $\mathrm{Col}(A)$.

- Recall: if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are (pairwise) orthogonal:

$$\boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \;=\; c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1$$

  Implies: the $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are independent (unless one is the zero vector)

Orthogonal bases

Definition 1. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthogonal basis** if the vectors are (pairwise) orthogonal.

Example 2. The standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$.

Example 3. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

Solution.

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$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

So this is an orthogonal basis.

Note that we do not need to check that the three vectors are independent. That follows from their orthogonality.

Example 4. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and that $\boldsymbol{w}$ is in V. Find $c_1, \ldots, c_n$ such that

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n.$$

Solution. Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned} \boldsymbol{v}_1 \cdot \boldsymbol{w} &= \boldsymbol{v}_1 \cdot (c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n) \\ &= c_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n \boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 \end{aligned}$$

Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$. In general, $c_j = \frac{\boldsymbol{v}_j \cdot \boldsymbol{w}}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}$.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

Example 5. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution.

$$\begin{aligned} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Definition 6. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthonormal basis** if the vectors are orthogonal and have length 1.

Example 7. The standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \boldsymbol{v}_j \cdot \boldsymbol{w}.$$

Example 8. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution. That's trivial, of course:

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But note that the coefficients are

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 7, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4.$$

Example 9. Is the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ orthonormal? If not, normalize the vectors to produce an orthonormal basis.

Solution.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = 1 \implies$ is already normalized: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The corresponding orthonormal basis is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 10. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

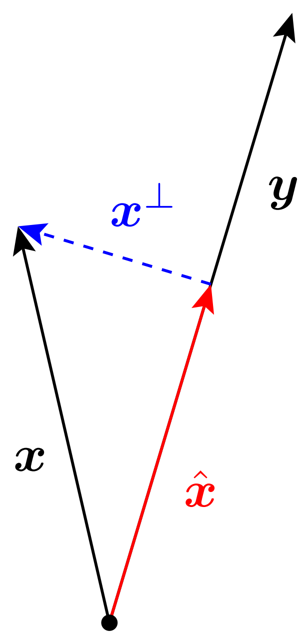
Solution.

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{-4}{\sqrt{2}}, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{10}{\sqrt{2}}, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4.$$

Hence, just as in Example 5:

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \frac{-4}{\sqrt{2}} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{\sqrt{2}} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthogonal projections



Definition 11. The **orthogonal projection** of vector $\boldsymbol{x}$ onto vector $\boldsymbol{y}$ is

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}} \boldsymbol{y}.$$

- The vector $\hat{\boldsymbol{x}}$ is the closest vector to $\boldsymbol{x}$, which is in $\operatorname{span}\{\boldsymbol{y}\}$.
- Characterized by: the "error" $\boldsymbol{x}^{\perp} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ is orthogonal to $\operatorname{span}\{\boldsymbol{y}\}$.
- To find the formula for $\hat{\boldsymbol{x}}$, start with $\hat{\boldsymbol{x}} = c\boldsymbol{y}$.

  $(\boldsymbol{x} - \hat{\boldsymbol{x}}) \cdot \boldsymbol{y} = (\boldsymbol{x} - c\boldsymbol{y}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} - c\boldsymbol{y} \cdot \boldsymbol{y} \overset{\text{wanted}}{=} 0$

  It follows that $c = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}}$.

$\boldsymbol{x}^{\perp}$ is also called the **component of $\boldsymbol{x}$ orthogonal to $\boldsymbol{y}$**.

Example 12. What is the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Solution.

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}} \boldsymbol{y} = \frac{-8 \cdot 3 + 4 \cdot 1}{3^2 + 1^2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$$

The component of $\boldsymbol{x}$ orthogonal to $\boldsymbol{y}$ is

$$\boldsymbol{x} - \hat{\boldsymbol{x}} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

(Note that, indeed $\begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ are orthogonal.)

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Example 13. What are the orthogonal projections of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Solution.

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ on $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$: $\frac{2\cdot 1 + 1\cdot(-1) + 1\cdot 0}{1^2 + (-1)^2 + 0^2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ on $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$: $\frac{2\cdot 1 + 1\cdot 1 + 1\cdot 0}{1^2 + 1^2 + 0^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ on $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: $\frac{2\cdot 0 + 1\cdot 0 + 1\cdot 1}{0^2 + 0^2 + 1^2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Note that these sum up to $\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$!

That's because the three vectors are an orthogonal basis for $\mathbb{R}^3$.

Recall: If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w}\cdot\boldsymbol{v}_j}{\boldsymbol{v}_j\cdot\boldsymbol{v}_j}.$$

$\rightsquigarrow$ $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector