Review

- $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if $\boldsymbol{v}\cdot\boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n = 0$.
  - This simple criterion is equivalent to Pythagoras' theorem.
  - Nonzero orthogonal vectors are independent.
- $\operatorname{Nul}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$, $\operatorname{Col}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}^T\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$
- *Fundamental Theorem of Linear Algebra*: (A an $m \times n$ matrix)
  - $\operatorname{Nul}(A)$ is orthogonal to $\operatorname{Col}(A^T)$. (both subspaces of $\mathbb{R}^n$)

    Moreover, $\underbrace{\dim \operatorname{Col}(A^T)}_{= r \text{ (rank of } A)} + \underbrace{\dim \operatorname{Nul}(A)}_{= n - r} = n$

    Hence, they are **orthogonal complements** in $\mathbb{R}^n$.
  - $\operatorname{Nul}(A^T)$ and $\operatorname{Col}(A)$ are orthogonal complements. (in $\mathbb{R}^m$)

$\operatorname{Nul}(A)$ is orthogonal to $\operatorname{Col}(A^T)$.

Why? Suppose that $\boldsymbol{x}$ is in $\operatorname{Nul}(A)$. That is, $A\boldsymbol{x} = \boldsymbol{0}$.
But think about what $A\boldsymbol{x} = \boldsymbol{0}$ means (row-column rule).
It means that the inner product of every row with $\boldsymbol{x}$ is zero.
But that implies that $\boldsymbol{x}$ is orthogonal to the row space.

Example 1. Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution. (FTLA, no thinking) In other words:

find the orthogonal complement of $\operatorname{Col}\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$.

FTLA: this is $\operatorname{Nul}\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}^T\right) = \operatorname{Nul}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$,

which has basis: $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

$\operatorname{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ are the vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

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Solution. (a little thinking) The FTLA is not magic!

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \boldsymbol{x} = 0 \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \boldsymbol{x} = 0 \iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \boldsymbol{x} \text{ in } \operatorname{Nul}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right)$$

This is the same null space we obtained from the FTLA.

Example 2. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = 2c \right\}$.

Find a basis for the orthogonal complement of V.

Solution. (FTLA, no thinking) We note that $V = \operatorname{Nul}([\ 1 \ \ 1 \ \ -2\ ])$.

FTLA: the orthogonal complement is $\operatorname{Col}([\ 1 \ \ 1 \ \ -2\ ]^T)$.

Basis for the orthogonal complement: $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Solution. (a little thinking) $a + b = 2c \iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0$.

So: V is actually defined as the orthogonal complement of $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

A new perspective on $A\boldsymbol{x} = \boldsymbol{b}$

$A\boldsymbol{x} = \boldsymbol{b}$ is solvable

$\iff$ $\boldsymbol{b}$ is in $\operatorname{Col}(A)$ ("direct" approach)

$\iff$ $\boldsymbol{b}$ is orthogonal to $\operatorname{Nul}(A^T)$ ("indirect" approach)

The indirect approach means: if $\boldsymbol{y}^T A = \boldsymbol{0}$ then $\boldsymbol{y}^T \boldsymbol{b} = 0$.

Example 3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

Solution. (old)

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 5 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

So, $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\iff -3b_1 + b_2 + b_3 = 0$.

Solution. (new) $A\boldsymbol{x} = \boldsymbol{b}$ solvable $\iff$ $\boldsymbol{b}$ orthogonal to $\mathrm{Nul}(A^T)$

to find $\mathrm{Nul}(A^T)$: $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

We conclude that $\mathrm{Nul}(A^T)$ has basis $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is solvable $\iff \boldsymbol{b} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$. As above!

Motivation

Example 4. Not all linear systems have solutions.

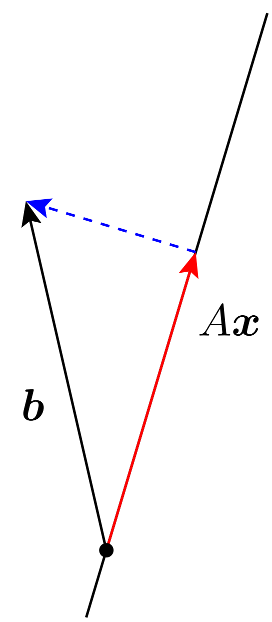
In fact, for many applications, data needs to be fitted and there is no hope for a perfect match.

For instance, $A\boldsymbol{x} = \boldsymbol{b}$ with

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

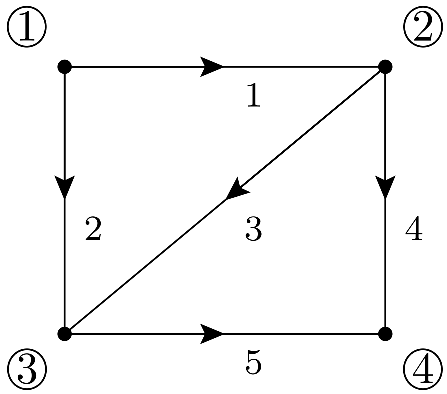
has no solution:

- $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in $\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- Instead of giving up, we want the $\boldsymbol{x}$ which makes $A\boldsymbol{x}$ and $\boldsymbol{b}$ as close as possible.



- Such $\boldsymbol{x}$ is characterized by $A\boldsymbol{x}$ being **orthogonal** to the error $\boldsymbol{b} - A\boldsymbol{x}$ (see picture!)

Application: directed graphs



- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- arrow indicates direction of flow
- no edges from a node to itself
- at most one edge between nodes

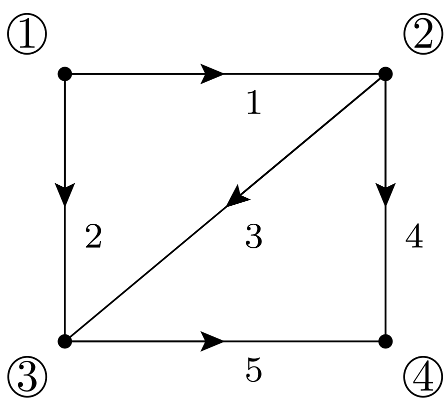
Definition 5. Let G be a graph with m edges and n nodes.

The **edge-node incidence matrix** of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j, \\ +1, & \text{if edge } i \text{ enters node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Example 6. Give the edge-node incidence matrix of our graph.

Solution.



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- each column represents a node
- each row represents an edge

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