Linear transformations

- A map $T: V \to W$ between vector spaces is **linear** if
  - $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
  - $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$
- Let A be an $m \times n$ matrix.

  $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$ is linear.
- $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ defined by $T(p(t)) = p'(t)$ is linear.
- The only linear maps $T: \mathbb{R} \to \mathbb{R}$ are $T(x) = \alpha x$.

  Recall that $T(0) = 0$ for linear maps.
- Linear maps $T: \mathbb{R}^2 \to \mathbb{R}$ are of the form $T\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x + \beta y$.

  For instance, $T(x, y) = xy$ is not linear: $T\begin{pmatrix} 2x \\ 2y \end{pmatrix} \neq 2T(x, y)$

Example 1. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

- What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?

  $\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

  $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix}$

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V.
A linear map $T: V \to W$ is determined by the values $T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)$.

Why?
Take any $\boldsymbol{v}$ in V.
Write $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}_n$. (Possible, because $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ spans V.)
By linearity of T,

$$T(\boldsymbol{v}) = T(c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}) = c_1T(\boldsymbol{x}_1) + \ldots + c_nT(\boldsymbol{x}_n).$$

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Important geometric examples

We consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$, which are defined by matrix multiplication, that is, by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

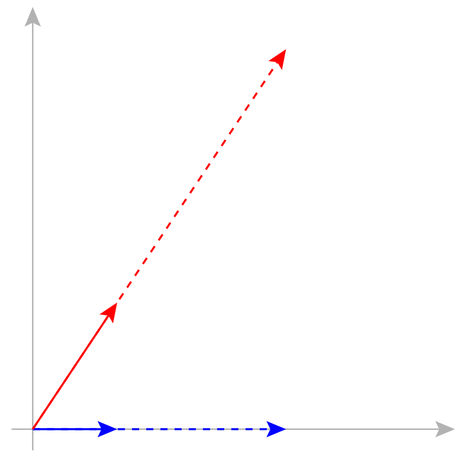
In fact: all linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\boldsymbol{x} \mapsto A\boldsymbol{x}$, for some matrix A.

Example 2.

The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

... gives the map $\boldsymbol{x} \mapsto c\boldsymbol{x}$, i.e.

... stretches every vector in $\mathbb{R}^2$ by the same factor c.

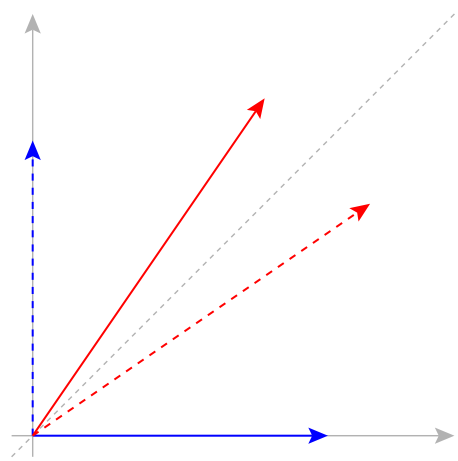


Example 3.

The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$, i.e.

... reflects every vector in $\mathbb{R}^2$ through the line $y = x$.

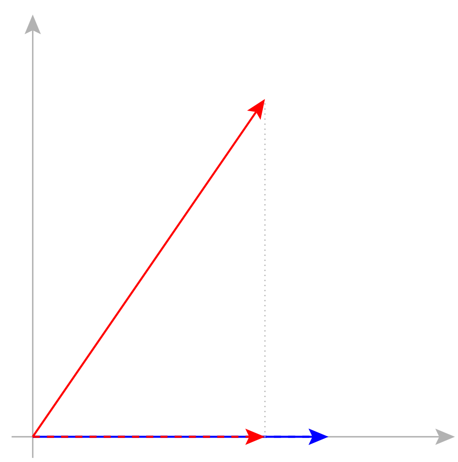


Example 4.

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$, i.e.

... projects every vector in $\mathbb{R}^2$ through onto the x-axis.



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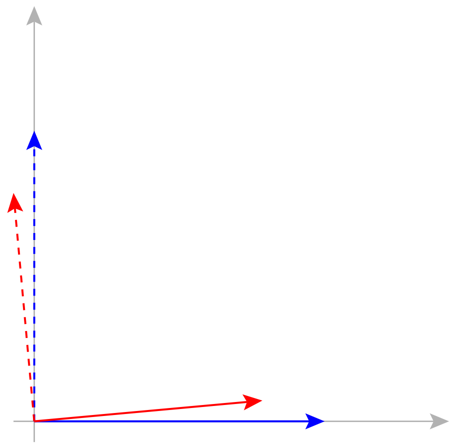
Example 5.

The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e.

... rotates every vector in $\mathbb{R}^2$ counter-clockwise by $90°$.



Representing linear maps by matrices

Definition 6. (From linear maps to matrices)

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ a basis for W.

The **matrix representing** T with respect to these bases

- has n columns (one for each of the $\boldsymbol{x}_j$),
- the j-th column has m entries $a_{1,j}, \ldots, a_{m,j}$ determined by

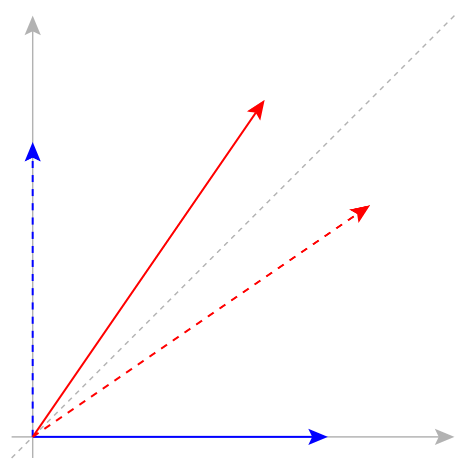
$$T(\boldsymbol{x}_j) = a_{1,j}\boldsymbol{y}_1 + \ldots + a_{m,j}\boldsymbol{y}_m.$$

Example 7.

Recall the map T given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$.

(reflects every vector in $\mathbb{R}^2$ through the line $y = x$)

- Which matrix A represents T with respect to the standard bases?
- Which matrix B represents T with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?



Solution.

- $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, $A = \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}$.

  $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  If a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix A with respect to the standard bases, then $T(\boldsymbol{x}) = A\boldsymbol{x}$.

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Matrix multiplication corresponds to function composition!

That is, if T_1, T_2 are represented by A_1, A_2, then $T_1(T_2(\boldsymbol{x})) = (A_1 A_2)\boldsymbol{x}$.

- $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, $B = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$.

  $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Example 8. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix B representing T with respect to the following bases?

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\boldsymbol{x}_2} \text{ for } \mathbb{R}^2, \qquad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3} \text{ for } \mathbb{R}^3.$$

Solution. This time:

$$\begin{aligned}
T(\boldsymbol{x}_1) &= T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix} \\
&= 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Longrightarrow B &= \begin{bmatrix} 5 & * \\ -3 & * \\ 5 & * \end{bmatrix} \\
T(\boldsymbol{x}_2) &= T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 11 \end{bmatrix} \\
&= 7\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Longrightarrow B &= \begin{bmatrix} 5 & 7 \\ -3 & -9 \\ 5 & 4 \end{bmatrix}
\end{aligned}$$

can you see it? otherwise: do it!

Tedious, even in this simple example! (But we can certainly do it.)

> A matrix representing T encodes in column j the coefficients of $T(\boldsymbol{x}_j)$ expressed as a linear combination of $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$.

Practice problems

Example 9. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map which rotates a vector counter-clockwise by angle θ.

- Which matrix A represents T with respect to the standard bases?
- Verify that $T(\boldsymbol{x}) = A\boldsymbol{x}$.

Solution. Only keep reading if you need a hint!

The first basis vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gets send to $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

Hence, the first column of A is ...

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