Review

- Elementary matrices performing row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}=\begin{bmatrix} a & b & c \\ d-2a & e-2b & f-2c \\ g & h & i \end{bmatrix}$$

- Gaussian elimination on A gives an LU decomposition $A=LU$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}=\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}$$

  U is the echelon form, and L records the inverse row operations we did.

- LU decomposition allows us to solve $A\boldsymbol{x}=\boldsymbol{b}$ for many $\boldsymbol{b}$.
- $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Already not so clear: $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab & -b & 1 \end{bmatrix}$

  Goal for today: invert these and any other matrices (if possible)

The inverse of a matrix

Example 1. The inverse of a real number a is denoted as a^{-1}. For instance, $7^{-1} = \frac{1}{7}$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$

In the context of $n \times n$ matrix multiplication, the role of 1 is taken by the $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Definition 2. An $n \times n$ matrix A is **invertible** if there is a matrix B such that

$$AB = BA = I_n.$$

In that case, B is the **inverse** of A and we write $A^{-1} = B$.

Example 3. We already saw that elementary matrices are **invertible**.

- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}$

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Note.

- The inverse of a matrix is unique. Why? So A^{-1} is well-defined.

  Assume B and C are both inverses of A. Then:

$$C = CI_n = CAB = I_nB = B$$

- Do not write $\frac{A}{B}$. Why?

  Because it is unclear whether it should mean AB^{-1} or $B^{-1}A$.

- If $AB = I$, then $BA = I$ (and so $A^{-1} = B$). Not easy to show at this stage.

Example 4. The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible. Why?

Solution.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Example 5. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided that $ad - bc \neq 0$.

Let's check that:

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = I_2$$

Note.

- A 1×1 matrix $[\ a\ ]$ is invertible $\Longleftrightarrow$ $a \neq 0$.
- A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Longleftrightarrow$ $ad - bc \neq 0$.

We will encounter the quantities on the right again when we discuss determinants.

Solving systems using matrix inverse

Theorem 6. Let A be invertible. Then the system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Proof. Multiply both sides of $A\boldsymbol{x} = \boldsymbol{b}$ with A^{-1} (from the left!). □

Example 7. Solve $\begin{array}{rrcl} -7x_1 & +3x_2 & = & 2 \\ 5x_1 & -2x_2 & = & 1 \end{array}$ using matrix inversion.

Solution. In matrix form $A\boldsymbol{x} = \boldsymbol{b}$, this system is

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Computing the inverse:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

Recall that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Hence, the solution is:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

Recipe for computing the inverse

To solve $A\boldsymbol{x} = \boldsymbol{b}$, we do row reduction on $\left[\begin{array}{c|c} A & \boldsymbol{b} \end{array}\right]$.

To solve $AX = I$, we do row reduction on $\left[\begin{array}{c|c} A & I \end{array}\right]$.

To compute A^{-1}: **Gauss–Jordan method**

- Form the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- Compute the reduced echelon form. (i.e. Gauss–Jordan elimination)
- If A is invertible, the result is of the form $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$.

Example 8. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution. By row reduction:

$$\left[\begin{array}{cc} A & I \end{array}\right] \rightsquigarrow \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Hence, $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$.

Example 9. Let's do the previous example step by step.

$$\left[\begin{array}{cc} A & I \end{array}\right] \qquad \rightsquigarrow \qquad \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{R2 \to R2 + \frac{3}{2}R1}{\rightsquigarrow} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{R1 \to \frac{1}{2}R1 \\ R2 \leftrightarrow R3}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

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Note. Here is another way to see why this algorithm works:

- Each row reduction corresponds to multiplying with an elementary matrix E:

$$[\ A\,|\,I\ ] \rightsquigarrow [\ E_1A\,|\,E_1I\ ] \rightsquigarrow [\ E_2E_1A\,|\,E_2E_1\ ] \rightsquigarrow \ldots$$

- So at each step:

$$[\ A\,|\,I\ ] \rightsquigarrow [\ FA\,|\,F\ ] \qquad \text{with } F = E_r \cdots E_2E_1$$

- If we manage to reduce $[\ A\,|\,I\ ]$ to $[\ I\,|\,F\ ]$, this means

$$FA = I \qquad \text{and hence } A^{-1} = F.$$

Some properties of matrix inverses

Theorem 10. Suppose A and B are invertible. Then:

- A^{-1} is invertible and $(A^{-1})^{-1} = A$.

  Why? Because $AA^{-1} = I$

- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

  Why? Because $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ (Recall that $(AB)^T = B^T A^T$.)

- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

  Why? Because $(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I$

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