Pre-lecture: the shocking state of our ignorance

Q: *How fast can we solve N linear equations in N unknowns?*

Estimated cost of Gaussian elimination:

$$\begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

- to create the zeros below the pivot:
  $\Longrightarrow$ on the order of N^2 operations
- if there is N pivots:
  $\Longrightarrow$ on the order of $N \cdot N^2 = N^3$ op's

- A more careful count places the cost at $\sim\frac{1}{3}N^3$ op's.
- For large N, it is only the N^3 that matters.

  It says that if $N \to 10N$ then we have to work 1000 times as hard.

That's not optimal! We can do better than Gaussian elimination:

- Strassen algorithm (1969): $N^{\log_2 7} = N^{2.807}$
- Coppersmith–Winograd algorithm (1990): $N^{2.375}$
- ... Stothers–Williams–Le Gall (2014): $N^{2.373}$

Is N^2 possible? We have no idea! (better is impossible; why?)

Good news for applications: (will see an example soon)

- Matrices typically have lots of structure and zeros

  which makes solving so much faster.

Organizational

- Help sessions in 441 AH: MW 4-6pm, TR 5-7pm

Review

- A system such as

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

  can be written in **vector** form as

$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

- The left-hand side is a **linear combination** of the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The row and column picture

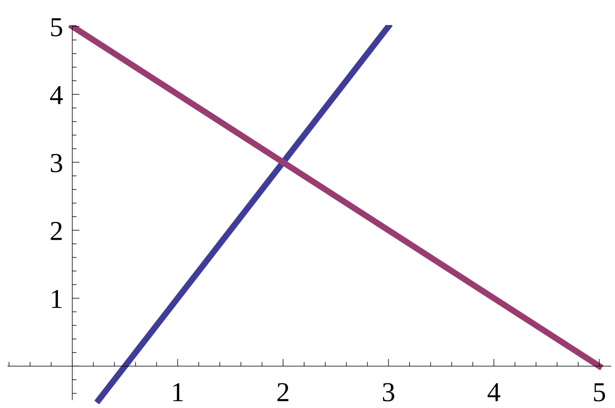
Example 1. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways. Here, there is a unique solution: $x = 2$, $y = 3$.

Row picture.

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.



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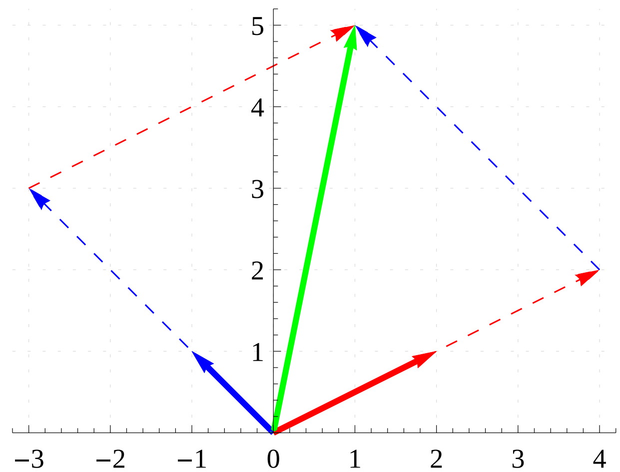
Column picture.

- The system can be written as

$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

- Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?
- $(2, \ 3)$ are the coefficients of the (only) such linear combination.



Example 2. Consider the vectors

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}.$$

Determine if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

Solution. Vector $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ if we can find weights x_1, x_2, x_3 such that:

$$x_1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

This vector equation corresponds to the linear system:

$$\begin{array}{rrrcr} x_1 & +4x_2 & +3x_3 & = & -1 \\ & +2x_2 & +6x_3 & = & 8 \\ 3x_1 & +14x_2 & +10x_3 & = & -5 \end{array}$$

Corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right]$$

Note that we are looking for a linear combination of the first three columns which

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produces the last column.

Such a combination exists $\iff$ the system is consistent.

Row reduction to echelon form:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 2 & 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Since this system is consistent, $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

[It is consistent, because there is no row of the form $[\, 0 \;\; 0 \;\; 0 \;\; b \,]$ with $b \neq 0$.]

Example 3. In the previous example, express $\boldsymbol{b}$ as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

Solution. The reduced echelon form is:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & -5 & -10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We read off the solution $x_1 = 1$, $x_2 = -2$, $x_3 = 2$, which yields

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}.$$

Summary

A vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_m\boldsymbol{a}_m = \boldsymbol{b}$$

has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} | & | & & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m & \boldsymbol{b} \\ | & | & & | & | \end{bmatrix}.$$

In particular, $\boldsymbol{b}$ can be generated by a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ if and only if this linear system is consistent.

The span of a set of vectors

Definition 4. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all their linear combinations. We denote it by $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

In other words, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all vectors of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m,$$

where $c_1, c_2, \ldots, c_m$ are scalars.

Example 5.

(a) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ geometrically.

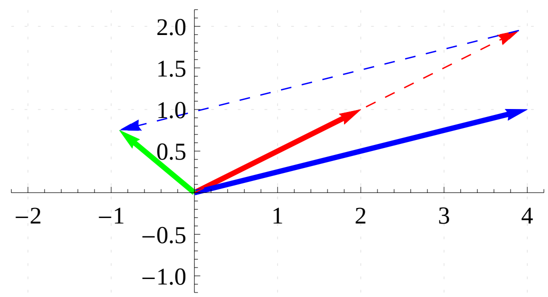
The span consists of all vectors of the form $\alpha \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

As points in $\mathbb{R}^2$, this is a line.

(b) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$ geometrically.

The span is all of $\mathbb{R}^2$, a plane.

That's because any vector in $\mathbb{R}^2$ can be written as $x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.



Let's show this without relying on our geometric intuition: let $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ any vector.

$$\left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 1 & 1 & b_2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 0 & -1 & b_2 - \frac{1}{2}b_1 \end{array}\right] \quad \text{is consistent}$$

Hence, $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(c) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}$ geometrically.

Note that $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence, the span is as in (a).

Again, we can also see this after row reduction: let $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ any vector.

$$\left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 1 & 2 & b_2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 0 & 0 & b_2 - \frac{1}{2}b_1 \end{array}\right] \quad \text{is not consistent for all } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the span of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ only if $b_2 - \frac{1}{2}b_1 = 0$ (i.e. $b_2 = \frac{1}{2}b_1$).

So the span consists of vectors $\begin{bmatrix} b_1 \\ \frac{1}{2}b_1 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$.

A single (nonzero) vector always spans a line, two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ usually span a plane but it could also be just a line (if $\boldsymbol{v}_2 = \alpha \boldsymbol{v}_1$).

We will come back to this when we discuss dimension and linear independence.

Example 6. Is $\operatorname{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$ a line or a plane?

Solution. The span is a plane unless, for some α,

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Looking at the first entry, $\alpha = 2$, but that does not work for the third entry. Hence, there is no such α. The span is a plane.

Example 7. Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}.$$

Is $\boldsymbol{b}$ in the plane spanned by the columns of A?

Solution. $\boldsymbol{b}$ in the plane spanned by the columns of A if and only if

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right]$$

is consistent.

To find out, we row reduce to an echelon form:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{array}\right]$$

From the last row, we see that the system is inconsistent. Hence, $\boldsymbol{b}$ is not in the plane spanned by the columns of A.

Conclusion and summary

- The **span** of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ is the set of all their **linear combinations**.
- Some vector $\boldsymbol{b}$ is in $\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$ if and only if there is a solution to the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m & \boldsymbol{b} \\ | & | & & | & | \end{array}\right].$$

  - Each solution corresponds to the weights in a linear combination of the $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ which gives $\boldsymbol{b}$.
  - This gives a second geometric way to think of linear systems!