Introduction to systems of linear equations

These slides are based on Section 1 in *Linear Algebra and its Applications* by David C. Lay.

Definition 1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b.$$

Example 2. Which of the following equations are linear?

- $4x_1 - 5x_2 + 2 = x_1$ — linear: $3x_1 - 5x_2 = -2$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$ — linear: $2x_1 + x_2 - x_3 = 2\sqrt{6}$
- $4x_1 - 6x_2 = x_1x_2$ — not linear: x_1x_2
- $x_2 = 2\sqrt{x_1} - 7$ — not linear: $\sqrt{x_1}$

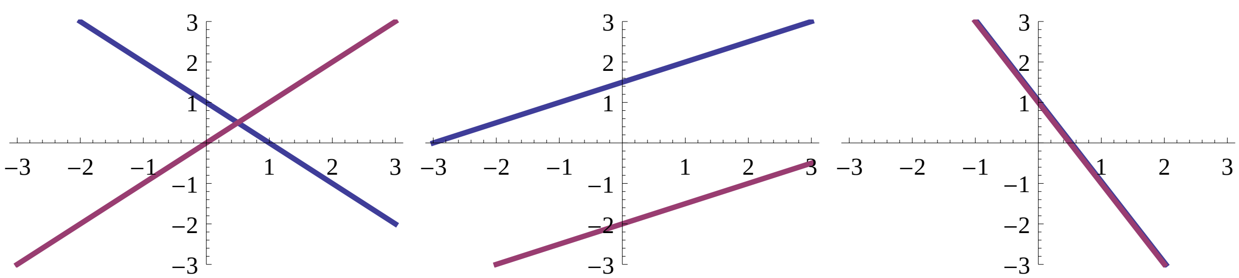
Definition 3.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.
- A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Example 4. (Two equations in two variables)

In each case, sketch the set of all solutions.

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ -x_1 & + & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ -4x_1 & - & 2x_2 & = & -2 \end{array}$$



Theorem 5. A linear system has either

- no solution, or
- one unique solution, or
- infinitely many solutions.

Definition 6. A system is **consistent** if a solution exists.

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How to solve systems of linear equations

Strategy: replace system with an equivalent system which is easier to solve

Definition 7. Linear systems are **equivalent** if they have the same set of solutions.

Example 8. To solve the first system from the previous example:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ -x_1 & + & x_2 & = & 0 \end{array} \quad \underset{\rightsquigarrow}{R2 \to R2+R1} \quad \begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ & & 2x_2 & = & 1 \end{array}$$

Once in this **triangular** form, we find the solutions by **back-substitution**:

$$x_2 = 1/2, \qquad x_1 = 1/2$$

Example 9. The same approach works for more complicated systems.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \downarrow \; R3 \to R3 + 4R1$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \downarrow \; R3 \to R3 + \frac{3}{2}R2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

By back-substitution:

$$x_3 = 3, \qquad x_2 = 16, \qquad x_1 = 29.$$

It is always a good idea to check our answer. Let us check that $(29, 16, 3)$ indeed solves the original system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{array}{rcrcrcr} 29 & - & 2\cdot 16 & + & 3 & \overset{\checkmark}{=} & 0 \\ & & 2\cdot 16 & - & 8\cdot 3 & \overset{\checkmark}{=} & 8 \\ -4\cdot 29 & + & 5\cdot 16 & + & 9\cdot 3 & \overset{\checkmark}{=} & -9 \end{array}$$

Matrix notation

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

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Definition 10. An **elementary row operation** is one of the following:

- **(replacement)** Add one row to a multiple of another row.
- **(interchange)** Interchange two rows.
- **(scaling)** Multiply all entries in a row by a nonzero constant.

Definition 11. Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 12. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example 13. Here is the previous example in matrix notation.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad \downarrow R3 \to R3 + 4R1$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad \downarrow R3 \to R3 + \tfrac{3}{2}R2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Instead of back-substitution, we can continue with row operations.

After $R2 \to R2 + 8R3$, $R1 \to R1 - R3$, we obtain:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & 2x_2 & & & = & 32 \\ & & & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Finally, $R1 \to R1 + R2$, $R2 \to \frac{1}{2}R2$ results in:

$$\begin{array}{rrrcr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We again find the solution $(x_1, x_2, x_3) = (29, 16, 3)$.

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Row reduction and echelon forms

Definition 14. A matrix is in **echelon form** (or **row echelon form**) if:

(1) Each leading entry (i.e. leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.

(2) All entries in a column below a leading entry are zero.

(3) All nonzero rows are above any rows of all zeros.

Example 15. Here is a representative matrix in echelon form.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($*$ stands for any value, and $\blacksquare$ for any nonzero value.)

Example 16. Are the following matrices in echelon form?

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ YES

(b) $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ NOPE (but it is after exchanging the first two rows)

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ YES

(d) $\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$ NO

Related and extra material

- In our textbook: parts of 1.1, 1.3, 2.2 (just pages 78 and 79)

  However, I would suggest waiting a bit before reading through these parts (say, until we covered things like matrix multiplication in class).

- Suggested practice exercise: 1, 4, 5, 10, 11 from Section 1.3

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