# Math 415 - Final Exam

Friday, December 12, 2014

Circle your section:

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|---|---|---|
| Philipp Hieronymi | 2pm | 3pm |
| Armin Straub | 9am | 11am |

Name:

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To be completed by the grader:

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*Good luck!*

## Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 180 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

## Important Note

- The collection of problems below is not representative of the final exam!
- The first three problems cover the material since the third midterm exam, and problems on the final exam on these topics will be of similar nature.
- Problems 4 and 5 are a good start to review the material we covered earlier; however, on the exam itself you should expect questions of the kind that we had on the previous midterms.
- In other words, **to prepare for the final, you need to also prepare our past midterm exams and practice exams**.
- In particular, a basic understanding of Fourier series or the ability to work with spaces of polynomials are expected.

**Problem 1.** Find a solution to the initial value problem (that is, differential equation plus initial condition)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{u}, \quad \boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Simplify your solution as far as possible.

**Solution.** The solution is $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. In order to compute $e^{At}$, we have to find eigenvalues and corresponding eigenvectors of $A$. We have:

$$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(\lambda(\lambda-1)-1)-(1-\lambda) = (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(-1-\lambda)(2-\lambda)$$

Hence, the eigenvalues of $A$ are 2, 1, and $-1$. For $\lambda = 2$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R2\to R2+R1, R3\to R3+R2, R1\to R1+R2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 1$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R3\to R3-R1, R2\to R2+R1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

For $\lambda = -1$:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R2\to R2-1/2R3, R1\to R1-R3, R1\to 1/2R1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Hence, $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. (columns of $P$ are linearly independent eigenvectors of $A$, and entries on the main diagonal of $D$ are the corresponding eigenvalues)

Note that the columns of $P$ are pairwise orthogonal so we can get $P^{-1}$ by dividing rows of $P^T$ by the square of the length of each row, i.e.:

$$P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

(Don't worry if you did not see this, and used Gauss–Jordan instead.) Since $A = PDP^{-1}$, we have $e^{At} = Pe^{Dt}P^{-1}$. Hence,

$$\begin{aligned} \boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0) &= Pe^{Dt}P^{-1}\boldsymbol{u}(0) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2t} + e^{t} \\ e^{2t} \\ e^{2t} - e^{t} \end{bmatrix} \end{aligned}$$

**Problem 2.** The processors of a supercomputer are inspected weekly in order to determine their condition. The condition of a processor can either be perfect, good, reasonable or bad.

A perfect processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good, and with probability 0.1 it is reasonable. A processor in good conditions is still good after one week with probability 0.6, reasonable with probability 0.2, and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The reparation takes one week, after which the processor is again in perfect condition.

In the steady state, what is percentage of processors in perfect condition?

**Solution.** We consider four states: perfect, good, reasonable, bad

The transition matrix is:

$$\begin{bmatrix} 0.7 & 0 & 0 & 1 \\ 0.2 & 0.6 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0 \\ 0 & 0.2 & 0.5 & 0 \end{bmatrix}$$

The steady state is the eigenvector corresponding to the eigenvalue 1 (with the extra condition that summation of the entries of the vector should be 1; since the states are percentages). We have:

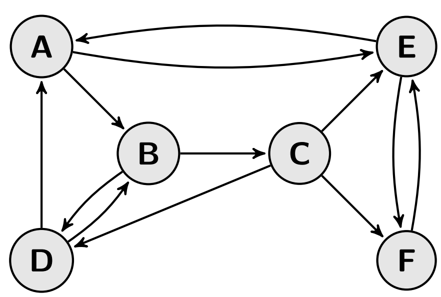
$$\begin{bmatrix} 0.7-1 & 0 & 0 & 1 \\ 0.2 & 0.6-1 & 0 & 0 \\ 0.1 & 0.2 & 0.5-1 & 0 \\ 0 & 0.2 & 0.5 & 0-1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0 & 0 & 1 \\ 0.2 & -0.4 & 0 & 0 \\ 0.1 & 0.2 & -0.5 & 0 \\ 0 & 0.2 & 0.5 & -1 \end{bmatrix} \xrightarrow{R2\to R2+2/3R1, R3\to R3+1/3R1}$$

$$\begin{bmatrix} -0.3 & 0 & 0 & 1 \\ 0 & -0.4 & 0 & \frac{2}{3} \\ 0 & 0.2 & -0.5 & \frac{1}{3} \\ 0 & 0.2 & 0.5 & -1 \end{bmatrix} \xrightarrow{R3\to R3+1/2R2, R4\to R4+1/2R2, R4\to R4+R3}$$

$$\begin{bmatrix} -0.3 & 0 & 0 & 1 \\ 0 & -0.4 & 0 & \frac{2}{3} \\ 0 & 0 & -0.5 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the eigenspace corresponding to eigenvalue 1 is span $\left\{\begin{bmatrix} \frac{10}{3} \\ \frac{5}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}\right\}$. Therefore, the steady state is $\begin{bmatrix} \frac{10}{22} \\ \frac{5}{22} \\ \frac{4}{22} \\ \frac{3}{22} \end{bmatrix}$. In particular, in the steady state (almost) 45% of processors are in perfect condition.

**Problem 3.** Determine the PageRank vector for the following system of webpages, and rank the webpages accordingly.




**Solution.** The transition matrix is:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

The steady state is the eigenvector corresponding to the eigenvalue 1 (with the extra condition that summation of the entries of the vector should be 1). We have:

$$\begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & -1 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & -1 \end{bmatrix} \xrightarrow{R2\to R2+1/2R1, R5\to R5+1/2R1} \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & -1 \end{bmatrix}$$

$$\xrightarrow{R3\to R3+1/2R2, R4\to R4+1/2R2} \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & \frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{5}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & -1 \end{bmatrix} \xrightarrow{R4\to R4+1/3R3, R5\to R5+1/3R3, R6\to R6+1/3R3}$$

$$\begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & \frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{3}{8} & -\frac{17}{24} & 1 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{13}{24} & -1 \end{bmatrix} \xrightarrow{R5\to R5+3/4R4, R6\to R6+1/4R4} \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & \frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{12} & 1 \\ 0 & 0 & 0 & 0 & \frac{7}{12} & -1 \end{bmatrix}$$

$$\xrightarrow{R6\to R6+R5} \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & \frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{12} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the eigenspace corresponding to eigenvalue 1 is span $\left\{ \begin{bmatrix} \frac{8}{7} \\ \frac{6}{7} \\ \frac{3}{7} \\ \frac{4}{7} \\ \frac{12}{7} \\ 1 \end{bmatrix} \right\}$. Therefore, the PageRank vector for the system is $\frac{1}{40} \left\{ \begin{bmatrix} 8 \\ 6 \\ 3 \\ 4 \\ 12 \\ 7 \end{bmatrix} \right\}$.

The corresponding ranking is E, A, F, B, D, C.

**Problem 4.** Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find bases for Nul($A$) and Col($A$).
(b) Determine the LU decomposition of $A$.
(c) Determine the inverse of $A$.
(d) What is the determinant of $A$?
(e) Determine the QR decomposition of $A$.
(f) Determine the eigenvalues of $A$ and find bases for the eigenspaces.
(g) Diagonalize $A$.

**Solution.** (a) We transform $A$ into (the row reduced) echelon form. We have:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 + R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since all columns are pivot columns, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for Col($A$) and the empty set is a basis for Nul($A$) (since Nul($A$) $= \{0\}$).

(b) First, we transform $A$ to echelon form (an upper triangular matrix) using upward row operations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 + R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

To get $L$, we have to apply the inverse of the row operations in the reverse order to $I$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 - R2, R2 \to R2 + R1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = L$$

(c) We have:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - R1, R3 \to R3 + R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R1 \to R1 + R2, R2 \to R2 - 1/2R3, R1 \to R1 - 1/2R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R2 \to -R2, R3 \to 1/2R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(d) We use row operations to transform $A$, into an upper triangular matrix, $B$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 + R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = B$$

Since the row operations that we used do not change the value of the determinant, we have $\det(A) = \det(B)$. Hence,

$$\det(A) = \det(B) = 1.(-1).2 = -2.$$

(e) We start with the columns of $A(=[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3])$ and we use Gram-Schmidt to find the columns of $Q(=[\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3])$:

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\|\begin{bmatrix}1\\1\\0\end{bmatrix}\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$$

and,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2-(\mathbf{q}_1\cdot\mathbf{v}_2)\mathbf{q}_1}{\|\mathbf{v}_2-(\mathbf{q}_1\cdot\mathbf{v}_2)\mathbf{q}_1\|} = \frac{\begin{bmatrix}1\\0\\1\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}}{\|\begin{bmatrix}1\\0\\1\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\|} = \frac{\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}{\|\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\|} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}$$

and,

$$\begin{aligned}\mathbf{q}_3 &= \frac{\mathbf{v}_3-(\mathbf{q}_1\cdot\mathbf{v}_3)\mathbf{q}_1-(\mathbf{q}_2\cdot\mathbf{v}_3)\mathbf{q}_2}{\|\mathbf{v}_3-(\mathbf{q}_1\cdot\mathbf{v}_3)\mathbf{q}_1-(\mathbf{q}_2\cdot\mathbf{v}_3)\mathbf{q}_2\|}\\
&= \frac{\begin{bmatrix}0\\1\\1\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}}{\|\begin{bmatrix}0\\1\\1\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}-(\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix})\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}\|}\\
&= \frac{\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}}{\|\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}\|} = \begin{bmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}\end{aligned}$$

Hence,

$$Q = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

Finally:

$$R = Q^TA = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}}\\-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\0 & 0 & \frac{2}{\sqrt{3}}\end{bmatrix}$$

(f) If we expand along the first row, we obtain:

$$\det\begin{bmatrix}1-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & 1-\lambda\end{bmatrix} = (1-\lambda)(\lambda(\lambda-1)-1)-(1-\lambda) = (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(-1-\lambda)(2-\lambda)$$

Hence, the eigenvalues of $A$ are 2,1, and $-1$. For $\lambda = 2$:

$$\begin{bmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{bmatrix} \xrightarrow{R2\to R2+R1, R3\to R3+R2, R1\to R1+R2} \begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

For $\lambda = 1$:

$$\begin{bmatrix}0 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & 0\end{bmatrix} \xrightarrow{R3\to R3-R1, R2\to R2+R1} \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

For $\lambda = -1$:

$$\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R2\to R2-1/2R3, R1\to R1-R3, R1\to 1/2R1} \begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$.

(g) From (f), $A = PDP^{-1}$ where $P = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$. (columns of $P$ are linearly independent eigenvectors of $A$, and entries on the main diagonal of $D$ are the corresponding eigenvalues)

Note that columns of $P$ are pairwise orthogonal so we can get $P^{-1}$ by dividing rows of $P^T$ by the square of the length of each row, i.e.:

$$P^{-1} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{bmatrix}$$

**Problem 5.** Consider $A = \begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}$.

(a) Find orthogonal bases for all four fundamental subspaces.

(b) Determine the projection matrices corresponding to orthogonal projection onto Col($A$) and Col($A^T$).

(c) Consider the linear function $T : \mathbb{R}^2 \to \mathbb{R}^3$, which maps $\boldsymbol{x}$ to $A\boldsymbol{x}$.

- Determine the matrix which represents $T$ with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

- Determine the matrix which represents $T$ with respect to the basis $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ for $\mathbb{R}^2$, and $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ for $\mathbb{R}^3$.

(d) Find the least squares solution to $A\boldsymbol{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

**Solution.** (a) We transform $A$ into echelon form:

$$\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix} \xrightarrow{R1\to R1-R2, R1\to R1-R3, R1\leftrightarrow R2, R2\leftrightarrow R3} \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

Both columns of $A$ are pivot columns so $\mathrm{Nul}(A) = \{0\}$ and the empty set is a basis for $\mathrm{Nul}(A)$; and a basis for $\mathrm{Col}(A)$ is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right\} =$

$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\right\}$ is an orthogonal basis for $\mathrm{Col}(A)$.

We transform $A^T$ into echelon form:

$$\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix} \xrightarrow{R2\to R2-R1, R2\to -R2, R1\to R1-R2} \begin{bmatrix}1&0&1\\0&1&-1\end{bmatrix}$$

Hence, $\mathrm{Nul}(A^T) = \mathrm{span}\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ is a basis for $\mathrm{Nul}(A^T)$. Since the first and the second column of $A$ are pivot columns, a basis for $\mathrm{Col}(A^T)$ is span $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}\begin{bmatrix}1\\1\end{bmatrix}\right\} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\right\}$ is an orthogonal basis for $\mathrm{Col}(A^T)$.

(b) Let $W = \mathrm{Col}(A)$. We have to find the orthogonal projection of elements of the standard basis onto $W$. The orthogonal projection of the first standard basis vector is:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

The orthogonal projection of the second standard basis vector is:

$$\begin{bmatrix}0\\1\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{bmatrix}$$

The orthogonal projection of the third standard basis vector is:

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{bmatrix}$$

Hence, the projection matrix is:

$$\begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

Also, since $\text{Col}(A^T) = \mathbb{R}^2$ the projection matrix corresponding to orthogonal projection onto $\text{Col}(A^T)$ is the $2 \times 2$ identity matrix.

(c)

- $$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$
  $$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

  Hence, the matrix $A$ which represents $T$ with respect to the standard bases is

  $$\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}.$$

- $$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix}$$
  $$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\1\end{bmatrix}$$

  Hence, the matrix $A$ which represents $T$ with respect to the given bases is

  $$\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}.$$

(d) We have to solve $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$:

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and,

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since,

$$\left[\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow{R1 \to R1-2R2, R2 \to R2+2/3R1} \left[\begin{array}{cc|c} 0 & -3 & -2 \\ 1 & 0 & \frac{2}{3} \end{array}\right]$$

we obtain,

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

## MULTIPLE CHOICE

(? questions, 2 points each)

**Instructions for multiple choice questions**

- No reason needs to be given. There is always exactly one correct answer.
- Enter your answer on the scantron sheet that is included with your exam.
  In addition, on your exam paper, circle the choices you made on the scantron sheet.
- Use a **number 2 pencil** to shade the bubbles completely and darkly.
- Do **NOT** cross out your mistakes, but rather erase them thoroughly before entering another answer.
- Before beginning, please code in your name, UIN, and netid in the appropriate places. In the **'Section' field on the scantron**, please enter

  000 if Armin Straub is your instructor,

  001 if Philipp Hieronymi is your instructor.

The actual exam will have multiple choice questions here.

The midterm exams as well as the practice exams have plenty of problems that you can (and should) look at again. Below are the short problems and multiple choice questions from the conflict exam of our midterms.

**Shorts 1.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute $A^T A$.

**Solution.**

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

**Shorts 2.** Let $A$ be a matrix such that, for every $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ x+y \\ 2x+z \end{bmatrix}$.

Then, what is $A$?

**Solution.** $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

**Shorts 3.** Let $C$ be a $3 \times 3$ matrix such that $C$ has three pivot columns, and let $\boldsymbol{d}$ be a vector in $\mathbb{R}^3$. Is it true that, if the equation $C\boldsymbol{x} = \boldsymbol{d}$ has a solution, then it has infinitely many solutions?

(a) True.
(b) False.
(c) Unable to determine.

**Solution.** (b), $C$ is invertible so for every $\boldsymbol{d}$, a vector in $\mathbb{R}^3$, $C\boldsymbol{x} = \boldsymbol{d}$ has a unique solution.

**Shorts 4.** Let
$$A = \begin{bmatrix} a-1 & a \\ a & a-1 \end{bmatrix}.$$
For which choice(s) of $a$ is the matrix $A$ *not* invertible?

**Solution.** We have:
$$\det(A) = (a-1)^2 - a^2 = -2a + 1 = 0 \Leftrightarrow a = \frac{1}{2}$$
$A$ is invertible if and only if $\det(A) \neq 0$. Hence, $A$ is not invertible if and only if $a = \frac{1}{2}$.

**Shorts 5.** Write down a $3 \times 3$-matrix that is not the zero matrix (i.e. the matrix whose entries are all zero) and is not invertible.

**Solution.** $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

**Shorts 6.** Let $W_1$ be the set of all $2 \times 2$-matrices $A$ such that $A$ is invertible, and let $W_2$ be the set of all $2 \times 2$-matrices $A$ such that $A^T = -A$. Are these sets subspaces of the vector space of all $2 \times 2$-matrices?

(a) Both $W_1$ and $W_2$ are subspaces.
(b) Only $W_1$ is a subspace.
(c) Only $W_2$ is a subspace.
(d) Neither $W_1$ nor $W_2$ are subspaces.

**Solution.** (c)

**Shorts 7.** Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Which of the following is true?

(a) $W$ is empty.
(b) $W$ is a line.
(c) $W$ is a plane.
(d) $W$ is all of $\mathbb{R}^3$.

**Solution.** (d)

**Shorts 8.** Let $H$ be a subspace of $\mathbb{R}^6$ with basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4, \boldsymbol{b}_5\}$. What is the dimension of $H$?

**Solution.** The dimension of $H$ is the number of vectors in a basis for $H$, i.e., 5.

**Shorts 9.** Which of the following collections of vectors is linearly independent?

$(a)$ $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$(b)$ $$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$(c)$ $$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$(d)$ $$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Solution.** (d)

**Shorts 10.** Let $A$ be an $4 \times 5$ matrix of rank 2. Is it possible to find two linearly independent vectors that are orthogonal to the null space of $A$? Is it possible to find two linearly independent vectors that are orthogonal to the left null space of $A$?

(a) Possible for both.
(b) Possible only for the column space.
(c) Possible only for the row space.
(d) Not possible in either case.
(e) Not enough information to decide.

**Solution.** (a)

**Shorts 11.** Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2, and let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(p(t)) = 3p(t) + 2p'(t).$$

Which matrix $A$ represents $T$ with respect to the standard bases?
(Recall that the standard basis for $\mathbb{P}_2$ is given by $1, t, t^2$.)

**Solution.**

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

**Shorts 12.** Let $V$ be the following subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 - 5x_3 = 0, \quad 10x_1 + 2x_2 - 4x_3 = 0 \right\}$$

What is the dimension of $V$?

**Solution.** $V = \text{Null}(\begin{bmatrix} 2 & -1 & -5 \\ 10 & 2 & -4 \end{bmatrix})$. Hence, the dimension of $V$ is 1.

**Shorts 13.** Let $a, b$ be in $\mathbb{R}$. Consider the three vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For which values of $a$ and $b$ are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ independent?

(a) $a = 0$ and $b = 1$
(b) $a \neq 0$ and $b \neq 1$
(c) $a = 0$ and $b \neq 1$
(d) $a \neq 0$ and $b = 1$

For which values of $a$ and $b$ does $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ have dimension 1?

(a) $a = 0$ and $b = 1$
(b) $a \neq 0$ and $b \neq 1$
(c) $a = 0$ and $b \neq 1$
(d) $a \neq 0$ and $b = 1$

**Solution.** (b),(a)

**Shorts 14.** What is the dimension of the orthogonal complement of

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}?$$

**Solution.** 1.

**Shorts 15.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

(a) What is the dimension of $\text{Nul}(A)$?
(b) What is the dimension of $\text{Col}(A)$?
(c) What is the dimension of $\text{Nul}(A^T)$?
(d) What is the dimension of $\text{Col}(A^T)$?

**Solution.** $A$ is of echelon form. We have:

(a) The dimension of $\text{Nul}(A)$ is the number non-pivot columns, i.e., 3.

(b) The dimension of Col($A$) is the number of pivot columns, i.e., 4.
(c) The dimension of Nul($A^T$) is the number of rows minus the number of pivot columns, i.e., 0.
(d) The dimension of Col($A^T$) is the number of pivot columns, i.e., 4.

**Shorts 16.** Suppose $v_1, v_2, v_3, v_4$ are four vectors in $\mathbb{R}^3$. Which of the following statements are correct for all such vectors?

(a) Any three of those vectors form a basis of $\mathbb{R}^3$,
(b) these vectors are linearly dependent,
(c) these vectors span $\mathbb{R}^3$,
(d) one of the vectors is a multiple of one of the other vectors.

**Solution.** (b).

**Shorts 17.** Consider the two matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which of the following is correct?

(a) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(b) $\text{Col}(A) = \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$
(c) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) = \text{Col}(B^T)$
(d) $\text{Col}(A) \neq \text{Col}(B)$ and $\text{Col}(A^T) \neq \text{Col}(B^T)$

**Solution.** (b)

**Shorts 18.** Let $W = \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$, and let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Suppose that $\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$ is in $W$ and $\boldsymbol{b}$ is orthogonal to $W$. Then:

(a) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(c) $\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

(d) none of the above

**Solution.** (a)

**Shorts 19.** Let $A$ be matrix with the property that $A^2 = A$. What is the best you can say about $\det(A)$?

(a) $\det A = 1$
(b) $\det A = \pm 1$
(c) $\det A \neq 0$
(d) $\det A = 1$ or $\det A = 0$

**Solution.** (d).

**Shorts 20.** If $A$ and $B$ are $3 \times 3$ matrices with $\det(A) = -2$ and $\det(B) = -1$. What is the determinant of $C = -2B^T BA$?

(a) 4
(b) $-8$
(c) 8
(d) $-16$
(e) 16

**Solution.** (e)

**Shorts 21.** Let $A, B$ be two $n \times n$-matrices. Consider the following two statements:

(S1) If $\det(A) = 0$, then two rows or two columns of $A$ are the same, or a row or a column of $A$ is zero.
(S2) If $AB \neq BA$, then $\det(AB) \neq \det(BA)$.

Then:

(a) Statement S1 and Statement S2 are correct.
(b) Only Statement S1 is correct.
(c) Only Statement S2 is correct.
(d) Neither Statement S1 nor Statement S2 is correct.

**Solution.** (d). (Note: if $n = 1$ then S1 is correct, but it is false for $n > 1$.)

**Shorts 22.** Which of the following choices for $a$ makes $\begin{bmatrix} 0 & 0 & 2 \\ 6 & a & 0 \\ 3a & 1 & 0 \end{bmatrix}$ invertible?

(a) any real number except $-\sqrt{2}$ and $\sqrt{2}$
(b) any real number except $-2$ and 2
(c) just $-\sqrt{2}$ and $\sqrt{2}$
(d) just $-2$ and 2

**Solution.** (a).

**Shorts 23.** Consider the following two statements:

(T1) If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are three orthonormal vectors, then the projection of $\boldsymbol{v_3}$ onto the span of $\boldsymbol{v_1}, \boldsymbol{v_2}$ is $\boldsymbol{v_3}$.

(T2) The Gram–Schmidt process produces from a linearly independent set $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ an orthonormal set $\{\boldsymbol{q_1}, \dots, \boldsymbol{q_n}\}$ with the property that for each $k \leq n$ the vectors $\{\boldsymbol{q_1}, \dots, \boldsymbol{q_k}\}$ span the same subspace as $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_k}\}$.

Then:

(a) Statement T1 and Statement T2 are correct.
(b) Only Statement T1 is correct.
(c) Only Statement T2 is correct.
(d) Neither Statement T1 nor Statement T2 is correct.

**Solution.** (c)

**Shorts 24.** Consider the vector space $V$ of all continuous functions $\mathbb{R} \to \mathbb{R}$, which are periodic with period $2\pi$, together with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

Let $f(t)$ be in $V$. Then the orthogonal projection of $f(t)$ onto the span of $\cos(4t)$ is

(a) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} \cos^2(4t)\,\mathrm{d}t} \cos(4t)$

(b) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} \cos^2(4t)\,\mathrm{d}t} f(t)$

(c) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t} f(t)$

(d) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t} \cos(4t)$

(e) none of the above

**Solution.** (a).

**Shorts 25.** Consider the space $\mathbb{P}^3$ of polynomials of degree up to 3, together with the inner product

$$\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t)\mathrm{d}t.$$

Which of the following sets of vectors is orthogonal with respect to this inner product?

(a) $\{1, t\}$
(b) $\{t, t^2\}$
(c) $\{2, -2t\}$
(d) none of the above

**Solution.** (d).

**Shorts 26.** Let $A$ be an $n \times n$ matrix. Consider the following two statements:

(U1) The matrix $8A$ has the same eigenvalues as $A$.
(U2) The matrix $8A$ has the same eigenvectors as $A$.

Then:

(a) Statement U1 and Statement U2 are correct.
(b) Only Statement U1 is correct.
(c) Only Statement U2 is correct.
(d) Neither Statement U1 nor Statement U2 is correct.

**Solution.** (c). (Note: if 0 is the only eigenvalue of $A$ then U1 is also correct, but it is not true in general.)

**Shorts 27.** Let $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto $W$, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto $W$. Then:

(a) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(e) $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

**Solution.** (c).